6. (a) Evaluate $\int_{-1}^{[illegible]} \frac{dx}{\sqrt{x^2+x+1}}$ giving your answer in the form $\ln a$, $a > 0$. (5 marks)

(b) Given the function f defined by

$$f(x) = \frac{|x-2|}{1-|x|}$$

(i) State the domain of f. (1 mark)

(ii) Show that $f(x) = \begin{cases} \frac{2-x}{1+x}, & x < 0 \\ \frac{2-x}{1-x}, & 0 \le x < 2 \\ \frac{x-2}{1-x}, & x \ge 2 \end{cases}$ (4 marks)

(iii) Investigate the continuity of f at $x = 2$. (2 marks)

7. (a) Find the greatest common divisor, d, of the integers 770 and 112. Express d in the form

$$d = 770x + 112y,\ x, y \in \mathbb{Z}.$$

Hence, find the general solution of the equation $770x + 112y = 28$. (4 marks)

(b) Prove by contradiction that if n^2 is even then n is even. (4 marks)

8. (a) Given that m and n are positive integers and that

$$I_{m,n} = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1-2x)^m (1+2x)^n \, dx,$$

show that

$$I_{m,n} = \left(\frac{m}{n+1}\right) I_{m-1,n+1},\ m \ge 1, n \ge 0.$$ (4 marks)

Evaluate $I_{2,4}$ (3 marks)

(b) Given that $f(x) = \frac{2x^2+2x+8}{x^2+4}$,

show that $f(x) = a + \frac{2x}{x^2+4}$ and find the value of a. (3 marks)

Show further that the point $(0, 2)$ is the centre of symmetry of the curve $y = f(x)$. (2 marks)

Given that $g(x) = f(x) - e^{-x}$, evaluate $\lim_{x\to\infty} g(x)$ (2 marks)

1. Given that $y = Axe^{2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4y = 8e^{2x},$$

Find

(i) the value of the constant A. (5 marks)

(ii) the general solution of the differential equation. (3 marks)

(iii) a particular solution of the differential equation for which $y = 1$, $\frac{dy}{dx} = -8$, when $x = 0$. (2 marks)

2. The position vectors of the points A, B and C are $(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, $(2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ and $(-\mathbf{i} - 3\mathbf{j})$ respectively, relative to the origin.

Find

(i) the Cartesian equation of the plane ABC. (6 marks)

(ii) the distance of the point $D(-2, -2, 5)$ from ABC. (3 marks)

3. (a) Determine whether or not the set of vectors $\mathbf{v}_1 = (2,-2,1)$, $\mathbf{v}_2 = (3,-5,4)$ and $\mathbf{v}_3 = (0,1,1)$ are linearly dependent. (4 marks)

(b) Determine the null space of the matrix $M = \begin{pmatrix} 1 & -7 \\ -3 & 21 \end{pmatrix}$. (4 marks)

4. (a) The polar curve C, has equation $C : r = \sqrt{2} + 2\sin\theta$.
Find the coordinates of the point that is furthest from the pole.
Find the tangents to the curve at the pole and sketch the curve. (5 marks)

(b) Given the function f defined by

$$f(x) = x - 1 + \frac{1}{e^x}$$

(i) Evaluate $\lim_{x \to -\infty} f(x)$ (2 marks)

(ii) Evaluate $\lim_{x \to +\infty} [f(x) - (x - 1)]$ and deduce the asymptote to the curve $y = f(x)$. (2 marks)

(iii) Investigate the relative position of the curve to its asymptotes and the variation of the curve by considering the monotonicity of f and present the information on a table. (2 marks)

(iv) Sketch the curve $y = f(x)$. (2 marks)

5. (a) Given the sequence U_n defined by

$$U_1 = 3,\ U_2 = 5 \text{ and } U_{n+2} - 3U_{n+1} + 2U_n = 0,$$

Prove by mathematical induction, or otherwise, that

$$U_n = 2^n + 1$$ (5 marks)

(b) Use De Moivre's theorem to show that

$$\sum_{r=0}^{\infty} \cos rx = \frac{1}{2}$$ (5 marks)

9. (a) Given the function $f(x) = x^3 - x^5 + \dfrac{\tan x}{\left(1+x^4\right)^2}$, determine the parity of f, and hence or otherwise,

evaluate $\int_{-5}^{5} f(x)\,dx$. **(3 marks)**

(b) Given that $(2\mathbb{Z}, +)$ and $(3\mathbb{Z}, +)$ are two groups and that $f : 2\mathbb{Z} \to 3\mathbb{Z}$ is a mapping defined as

$$f(x) = \frac{3}{2}x.$$

Show that f is a group homomorphism and determine whether or not f is an isomorphism. **(6 marks)**

10. The tangent at any point $P\left(cp, \dfrac{c}{p}\right)$ on the curve with equation $x = ct$ and $y = \dfrac{c}{t}$, where c is a constant,

meets the x – axis and the y – axis at R and S respectively.

Find the coordinates of R and S. (5 marks)

Show that P is the midpoint of RS. (2 marks)