

1. Derive Taylor's series solution, in ascending powers of x , up to and including the term in x^6 , of the differential equation.

$$\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2y = 0,$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 0$.

Use this series to estimate

$$\int_0^{\frac{1}{2}} y \, dx,$$

giving your answer to 3 decimal places.

Verify that $y = e^{-x^2}$ is a solution of the given differential equation and that it satisfies the given boundary conditions.

Use Simpson's rule with three ordinates to estimate.

$$\int_0^{\frac{1}{2}} e^{-x^2} \, dx,$$

giving your answer to 3 decimal places.

2. A bead A, of mass m , is threaded on a smooth circular wire, of radius a and center O. The wire is held in a vertical plane and A is projected along the wire from the lowest point B of the wire with speed U. Given that A just reaches the highest point of the wire, find U.

Prove that, at time t after project,

$$\frac{d\theta}{dt} = 2\sqrt{\left(\frac{a}{g}\right)} \cos\left(\frac{\theta}{2}\right).$$

Show that the time taken for A to reach the position, where OA is horizontal, is

$$\sqrt{\left(\frac{a}{g}\right)} \ln(1 + \sqrt{2}).$$

3. A particle, P of unit mass, moves on a plane curve. At time t , P is at the point with polar coordinates (r, θ) , referred to the pole O.

Derive expressions for the components of the acceleration of P along and perpendicular to the radius vector OP.

The radial and transverse components of velocity of a particle moving in a plane at time t , are $3\lambda r$ and $\lambda\sqrt{5kr}$, respectively, where λ and k are positive constants. Show that at time, t , the radial component of its acceleration is $\lambda^2(9r - 5k)$.

Find the transverse component of the acceleration of the particle in terms of λ , k and r .

Given that when $t = 0$, $r = 5k$ and $\theta = 0$, show that the polar equation of the path of the particle is

$$r = \frac{20k}{(2 - 3\theta)^2}$$

4. Show, by integration, that the moment of inertia of a thin uniform circular disc of mass m and radius a about an axis through its center O and perpendicular to its plane is $\frac{1}{2}ma^2$.

The disc can rotate in a vertical plane about an axis perpendicular to the plane of the disc through a point A on its circumference. It starts from rest when AO is horizontal. At time t , AO makes an angle θ with the horizontal.

Show that

(a) $3a \left(\frac{d\theta}{dt}\right)^2 = 4g \sin\theta,$

(b) the magnitudes of the horizontal and vertical components of the force exerted on the axis by the disc are $mg \sin 2\theta$ and $\frac{1}{3}mg(4 - 3 \cos 2\theta)$, respectively.

5. (a) Two smooth spheres of mass $3m$ and $2m$ have velocities $(-12\mathbf{i} + 8\mathbf{j})\mathbf{u}$ and $(5\mathbf{i} + 12\mathbf{j})\mathbf{u}$, respectively, where \mathbf{u} is a constant. The spheres collide with their line of centers parallel to the vector \mathbf{j} . Given that the coefficient of restitution between the spheres is $\frac{1}{4}$, find the loss in kinetic energy during the impact.
- (b) A sphere A, moving on a smooth horizontal table, strikes another sphere B, of the same size and mass, which is at rest on the table. The directions of motion of A before and after impact make angles 15° and 30° , respectively, with the line of centres. Show that the coefficient of restitution between A and B is e where $e = 1 - 2\sqrt{3} \tan 15^\circ$.

6. The masses of articles produced in a particular factory are normally distributed with mean μ and standard deviation σ . Given that 5% of the articles have a mass greater than 85 g and 10% have a mass less than 25 g, find the values of μ and σ , correct to 3 significant figures.

7. (a) A particle, P, of mass m , moves in a horizontal straight line and is acted upon by a frictional force of magnitude $mk\mathbf{u}$ and a retarding force of magnitude mkv , where v is the speed of P and k, u are positive constants. Given that $v = u$ when $t = 0$, show that P comes to rest after time $T = \frac{\ln 2}{k}$ and after traveling a distance $X = \frac{u}{k}(1 - \ln 2)$.

- (b) A particle of mass m moves along the x -axis under a restoring force $25mw^2x$ and a resisting force $6mwv$, where w is a constant and x, v are the displacement and speed, respectively, of the particle after time t . Show that

$$\frac{d^2x}{dt^2} + 6w\frac{dx}{dt} + 25w^2x = 0.$$

Given that $x = 0, \frac{dx}{dt} = u$ at $t = 0$, show further that

$$4wx = ue^{3wt} \sin(4wt).$$

8. Forces $\mathbf{F}_1 = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\text{ N}$, $\mathbf{F}_2 = (4\mathbf{i} + \mathbf{j} + 5\mathbf{k})\text{ N}$ and $\mathbf{F}_3 = (-6\mathbf{i} - 8\mathbf{k})\text{ N}$ act respectively through points with position vectors $\mathbf{r}_1 = (\mathbf{i} - 2\mathbf{j} - \mathbf{k})\text{ m}$, $\mathbf{r}_2 = (7\mathbf{i} - c\mathbf{j} + 7\mathbf{k})\text{ m}$, and $\mathbf{r}_3 = (3\mathbf{i} - 3\mathbf{j} + \mathbf{k})\text{ m}$, where c is a constant. Given that the lines of action of \mathbf{F}_1 and \mathbf{F}_2 meet at a point, find the value of c and determine a vector equation of the line of action of the resultant of the forces \mathbf{F}_1 and \mathbf{F}_2 .

Show that the system of forces $\mathbf{F}_1, \mathbf{F}_2$ and \mathbf{F}_3 is equivalent to a couple and find the magnitude of the couple. The force \mathbf{F}_3 is now replaced with a force \mathbf{F}_4 so that $\mathbf{F}_1, \mathbf{F}_2$ and \mathbf{F}_4 are in equilibrium. Find a vector equation of the line of action of \mathbf{F}_4 .