1. (i) Given that $1 + i\sqrt{3}$ is a root of the equation

$$z^3 + z^2 + az + b = 0,$$

find the values of the real numbers a and b. Hence, solve this equation.

(ii) Given that z_1 and z_2 are two complex numbers, show geometrically or otherwise that

(a) $|z_1 + z_2| \geq |z_1 - z_2|$, and

(b) $|z_1 - z_2| \geq |z_1| - |z_2|$.

Hence, find the least and greatest values of $|z_1 + z_2|$, when $z_1 = -3 + 4i$ and $|z_2| = 10$.

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2. (i) Express f(x), where, $f(x) \equiv \dfrac{x^2}{(x-1)^2(x^2+1)}$ in partial fractions.

Hence, find $\int f(x)\,dx$.

(ii) Define cosechy in terms of e^y.

Given that $x > 0$, prove that

$$\text{arcosech}\, x = \ln\left(\frac{1+\sqrt{(x^2+1)}}{x}\right).$$

Solve the equation arcosech $(2x) + \ln x = \ln 5$

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3. Given the matrix M, where

$$M = \begin{pmatrix} 2 & k & 2 \\ 6 & k & 2 \\ 4 & -3 & k \end{pmatrix},$$

(i) find the real values of k for which M is not invertible;

(ii) for $k = 0$, find

(a) the image of the line
$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$, under the transformation with matrix M,

(b) M^{-1},

(c) the point whose image is (-12, 0, 24), under the transformation with matrix M.

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4. (i) Sketch, using the same axes, the curves $y^2 = 4x$ and $y^2 = x^3$.
Shade the regions for which $(y^2 - 4x)(y^2 - x^3) \leq 0$.

(ii) Show that the polar equation of the curve C with Cartesian equation $(x^2 + y^2)^2 = 2a^2xy$, $a > 0$ is $r^2 = a^2 \sin 2\theta$.
Sketch the curve C, showing clearly the tangents at the pole. Verify that the tangent to C at the point where $\theta = \frac{\pi}{3}$, is parallel to the initial line.

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5. (i) Show that the arc length of the curve $ay^2 = x^3$, for $0 \le x \le \frac{7a}{3}$ is $\frac{13a}{3}$.

(ii) Given that

$x = a \sin 2t$, $y = a \cos 2t$, find the mean value of y in the interval $0 \le t \le \frac{\pi}{4}$,

(a) with respect to t,
(b) with respect to x.

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6. Find the equation of the tangent and the normal at the point. P$(2a + 2at, \frac{at^2}{2})$ to the parabola $(x - 2a)^2 = 2ay$.

The tangent and the normal at P cut the x – axis at T and N respectively.

Prove that $\frac{PT^2}{TN} = at$.

Find the coordinates of the point Q at which the normal at P intersects the parobola again.

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7. (i) A particular integral of the differential equation

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 2\sin 2t$$

is A sin2t + B cos2t. Find the values of the real constants A and B. Hence, solve this differential equation given that when t = 0, $x = 0$ and $\frac{dx}{dt} = 1$.

(ii) The function f is defined on the set of real numbers by $f(x) = x + [x]$,
where $[x]$ means the greatest integer less than or equal to x. Sketch the graph of f for $0 \le x < 2$.

Evaluate $\int_0^2 f(x)\, dx$.

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8. (i) The points A, B, C have position vectors **a**, **b**, **c**, respectively relative to the origin O, where
a = **i** + 2**j** + 3**k**, **b** = 2**i** – 4**k**, **c** = 5**i** – **j** – 3**k**.

Find,
(a) the vector equation of the plane ABC in the form **r.n** = p,
(b) the area of triangle ABC,
(c) the perpendicular distance of the point C from the line AB,
(d) the volume of the tetrahedron OABC.

(ii) Evaluate $\int_0^1 \sinh^{-1} 2x\,dx$, giving your answer to 4 decimal places.

Turn over

9. (i) Given that every element x of a group $(H, *)$ satisfies the relation $x * x = e$, where e is the identity element, prove that $(H, *)$ is a commutative group.

(ii) The elements of the set G are ordered pairs (a,b) in $Z_2 \times Z_3$, where $Z_2 = \{0,1\}$ and $Z_3 = \{0,1,2\}$.
Write down the six elements of G.
A binary operation $*$ on G is given by $(a,b) * (c,d) = (a +_2 c, b +_3 d)$ where $+_2$ and $+_3$ denote addition modulo 2 and addition modulo 3, respectively.

Show that $(G, *)$ forms a commutative group.
[You may assume associativity.]