1. (a) Solve the differential equation

$$x\frac{dy}{dx} - y = x^2 \cos x,$$

given that $y = 0$ when $x = \pi$. **(5 marks)**

(b) Find the constants A and B such that Acos2x + Bsin2x is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2\sin 2x.$$

Hence, obtain the general solution of the differential equation.

Find, also, the solution for which $y = 3$ and $\frac{dy}{dx} = 2$ when $x = 0$. **(12 marks)**

---

2. (a) Express $f(x) = \frac{x^2}{(x^2+1)^2}$ in partial fractions.

Hence, using the substitution $x = \tan\theta$ or otherwise, prove that

$$\int_0^1 f(x)dx = \tfrac{1}{8}(\pi - 2).$$ **(10 marks)**

(b) Given that $I_n = \int_1^{e^2} (\ln x)^n dx$, show that

$$I_n = 2^n e^2 - nI_{n-1}.$$

Hence, evaluate I_3. **(7 marks)**

---

3. (a) When x^5 and higher powers are neglected,

$$\ln\left(\frac{1-\sinh x}{1+x}\right) \approx ax + bx^3 + cx^4.$$

Find the values of the real constants a, b and c.
State the range of values of x for which the expansion is valid. **(10 marks)**

(b) Find the Maclaurin series expansion of $\cos x^2$ as far as the term in x^8.
Show that the general term, U_n, of this expansion can be written as

$$U_n = \frac{(-1)^n x^{4n}}{(2n)!}.$$

Hence, show that the series is convergent for all real values of x. **(7 marks)**

---

4. (a) The point P in the Argand diagram represents the complex number z, and Q represents the complex number ω, where $\omega = \frac{i}{z-1}$.

Given that P lies on the circle with centre at the origin and radius 1 unit,

(i) prove that Q lies on the curve $|\omega| = |\omega + i|$.

(ii) sketch the locus represented by $|\omega| = |\omega + i|$. **(5 marks)**

(b) Find the roots of the equation $(z-4)^3 = 8i$, giving your answer in the form $a + bi$, where a and b are real numbers.
Indicate, on an Argand diagram, the points A, B, C representing these roots. Find the area of triangle ABC. **(12 marks)**

---

5. Show that the length of the curve $8(y + \ln x) = x^2$ between $x = 1$ and $x = e$ is

$$\tfrac{1}{8}(7 + e^2).$$ **(7 marks)**

Find the area of the surface of revolution obtained by rotating this curve through 2π radians about the X-axis.
Using a theorem of Pappus, find the y coordinate of the centroid of this curve. **(10 marks)**

$\frac{e^x+e^{-x}}{2}$ $-4-3.$

$\cos u$
$\sin u$

$-\frac{(-\sin^2 x+\cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$

2\. (a) Find the real values of x for which

$$8\cosh x + 4\sinh x = 7,$$

giving your answers in terms of natural logarithms. **(5 marks)**

(b) Use the definition of $\coth x$ in terms of exponential functions to prove that

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ x^2 > 1$$

$\cos^2 - \sin^2 x = 1.$

$\coth^2 x - 1 = \operatorname{cosech}^2 x.$

A function f is defined by

$$f(x) = \coth^{-1}\left(\frac{x}{2}\right),\ x^2 > 4.$$ **(4 marks)**

(i) Show that $f'(x) = -\frac{2}{x^2-4}$. **(4 marks)**

(ii) Expand $f(x)$ as a series in ascending powers of $\frac{1}{x}$ as far the term in $\frac{1}{x^5}$. **(4 marks)**

7\. (a) Given the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},$$

$\frac{1/x}{2}.$

find the values(s) of λ for which

$|A - \lambda I| = 0$, where I is a unit matrix. **(5 marks)**

$\frac{1}{x} \times \frac{2}{2} = \frac{1}{2x}.$

(b) Show that the transformation T, represented by the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & -2 & -7 \end{pmatrix},$$

$\frac{-2}{\frac{1}{x^2}-4}$ $\frac{-2x^2}{1-4x^2}$

maps the whole space onto the plane $x - 2y + z = 0$. **(4 marks)**

Find the image under this transformation of

(i) the line $x = -y = \frac{z-1}{2}$, **(4 marks)**

(ii) the plane $x - y - z = 0$. **(4 marks)**

8\. (a) It is given that x is an element of a group G with identity element e and that $x^3 = x$.
Show that $x^2 = e$. **(4 marks)**

(b) Consider the groups G_1, G_2, G_3, where

$$G_1 : (\{1,3,7,9\}, \times_{10}).$$
$$G_2 : (\{1,5,7,11\}, \times_{12}),$$
$$G_3 : (\{1,3,5,7\}, \times_8),$$

where $\times_n$ means multiplication modulo n.

(i) Draw up group tables for G_1, G_2, G_3. **(5 marks)**

(ii) Find which of the two groups are isomorphic and write down an isomorphism between them. **(3 marks)**

(iii) Solve the equation $x^3 = x$ in each of the three groups. **(5 marks)**

9\. The lines L_1 and L_2 are given by

L_1: $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$,

L_2: $\mathbf{r} = 6\mathbf{i} - \mathbf{j} + \mathbf{k} + \mu(-2\mathbf{i} + \mathbf{k})$.

The plane π contains the lines L_1 and L_2.

Find,

(a) the position vector of the point of intersection of L_1 and L_2; **(6 marks)**

(b) a vector normal to the plane π; **(2 marks)**

(c) a Cartesian equation of the plane π; **(2 marks)**

(d) the distance of the point (3, -1, 4) from the plane π; **(2marks)**

(e) the position vector of the point of intersection of the line

$$\frac{x-1}{1} = \frac{y+2}{-3} = \frac{z-2}{2}$$ and the plane π. **(5 marks)**