JUNE 2002

(i) State the conditions under which a system of coplanar forces can reduce to a single force.
Three coplanar forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ act through the points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, respectively, where

$$\mathbf{F}_1 = (\mathbf{j} - \mathbf{k})\text{N}, \quad \mathbf{r}_1 = (2\mathbf{i} + \mathbf{k})\text{m},$$
$$\mathbf{F}_2 = (3\mathbf{i} - \mathbf{j} + \mathbf{k})\text{N}, \quad \mathbf{r}_2 = (5\mathbf{i} + \mathbf{k})\text{m},$$
$$\mathbf{F}_3 = (\mathbf{i} - 2\mathbf{j})\text{N}, \quad \mathbf{r}_3 = (2\mathbf{i} - \mathbf{j})\text{m}.$$

Prove that these forces are equivalent to a single force $\mathbf{F}$, and find the equation of the line of action of $\mathbf{F}$.

(ii) A force $\mathbf{F}$ of magnitude $6\sqrt{6}$ N acts parallel to the vector $(-5\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$m. A bead moves along a smooth straight wire from the point A to the point B, where $\mathbf{OA} = (3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$m and $\mathbf{OB} = (\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})$m, respectively, under the action of $\mathbf{F}$. Find the work done by $\mathbf{F}$.

(i) Two substances A and B react with each other to form a third substance C. One unit of A reacts with two units of B to give one unit of C. The number of units of C which have been formed at time t is x. The rate of formation of C is proportional to the product of the number of units of A and the number of units of B present at time t. At time t = 0, there are $2a$ units of A, $3a$ units of B and none of C. Assuming that x is a continuous variable, show that $\frac{dx}{dt} = k(2a - x)(3a - 2x)$, where k is a positive constant.

By solving this differential equation, show also, that $\ln\left(\frac{6a - 3x}{6a - 4x}\right) = kat.$

Find the number of units of C present at time $t = \frac{1}{ka}\ln\left(\frac{3}{2}\right)$.

(ii) Solve the vector differential equation $\frac{d\mathbf{r}}{dt} = 2\mathbf{r}$, given that $\mathbf{r}.\mathbf{i} = 1$ and $\mathbf{r} \times \mathbf{i} = 2\mathbf{j} + \mathbf{k}$ when $t = 0$.

One end of a light inextensible string of length a is attached to a fixed point O. A particle of mass $2m$ is attached to the other end of the string. Initially P is held at a point distant a vertically above O. It is then projected horizontally with speed $\sqrt{(2ag)}$. Show that the string is not slack initially.

Find the speed of P and the tension in the string when OP is inclined at an angle $\frac{\pi}{3}$ to the upward vertical through O. The equation of motion of P is $a\left(\frac{d\theta}{dt}\right)^2 = 2g(2 - \cos\theta)$.

Show that the time T that elapses before P first passes through the point B, which is vertically below O, is given by

$$T = \sqrt{\left(\frac{a}{2g}\right)} \int_0^{\pi} \frac{1}{\sqrt{(2 - \cos\theta)}} d\theta$$

Given that $a = 1\,m, g = 9.8\,ms^{-2}$, use Simpson's rule with 5 ordinates to estimate the value of T, in seconds, giving your answer to 2 significant figures.

Using the Taylor series expansion for y = f(x) about x_n, or otherwise, show that if $(x_{n-1}, y_{n-1}), (x_n, y_n), (x_{n+1}, y_{n+1})$ are three successive points on the curve y = f(x), then $y_{n+1} \approx y_{n-1} + 2h\left(\frac{dy}{dx}\right)_n$, where $h = (x_n - x_{n-1}) = (x_{n+1} - x_n)$.

Given that $(2x^2 + 2)\frac{dy}{dx} = y$ and that y = 1 when x = 0, show that the series solution of this differential equation in ascending powers of x as far as and including the term in x^3 is $1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{7}{24}x^3$.

Using the given approximation and a step length of 0.1, calculate the value of y when x = 0.2, giving your answer to 4 decimal places.

A particle P of mass m is suspended from a fixed point O, by a light elastic string of natural length a and modulus $4mg$. The particle released from rest from a point at a distance l vertically below its equilibrium position. The motion takes place in a medium, which offers a resistance to motion of magnitude $2mv\sqrt{\frac{g}{a}}$, where v is the speed of the particle. At time t the displacement of P below its equilibrium position is x. Show that so long as the string remains taut, $\ddot{x} + 2k\dot{x} + 4k^2x = 0$, where $k = \sqrt{\frac{g}{a}}$.

Hence, show that $x = le^{-kt}(\cos kt\sqrt{2} + \frac{1}{\sqrt{3}}\sin kt\sqrt{3})$.

Find the speed of P as it passes through the equilibrium position for the first time.

(i) A smooth sphere A, of mass m, is at rest on a smooth horizontal table. A second sphere B, of same mass and radius as A, is moving with speed u towards A and impinges obliquely on it. Just before impact, the direction of motion of B makes an angle $\frac{\pi}{3}$ with their line of centres. The coefficient of restitution between the spheres is e. Show that the kinetic energy lost during this impact is $\frac{1}{16}mu^2(1-e^2)$.

(ii) The polar coordinates of a particle P, moving in a plane are (r,θ).
At time t, the radial and transverse components of the velocity of P are λr and $\lambda\sqrt{(ar)}$, respectively, where λ and a are positive constants.
Determine the transverse component of the acceleration of P in terms of λ, a and r.
Given that $r=a$ and $\theta=0$ when t = 0, show that P moves on the curve with polar equation

$$r=\frac{4a}{(2-\theta)^2}, 0\leq\theta<2.$$

A uniform rod AB has mass $3m$ and length $2a$. Show that the moment of inertia of the rod about an axis through A and perpendicular to the rod is $4ma^2$. The rod is smoothly hinged about a horizontal axis through A. It is released from rest when in a horizontal position. Show that when the rod makes an angle θ with the downward vertical, its angular velocity is given by

$$8a\left(\frac{d\theta}{dt}\right)^2=3(1+3\cos\theta)g.$$

Find an expression for the magnitude of the reaction on the axis.
When the rod reaches a vertical position, the end B collides and adheres to a small ball of mass m, which is moving horizontally in the opposite direction with, speed $4\Omega a$ where Ω is the angular speed of the rod at the instant when the rod is vertical. Show that the angular speed of the composite body immediately after impact is $\frac{1}{2}\Omega$.

(i) A random variable X takes all positive integral values with

$$P(X=x)=\frac{ke^{-\lambda x}\lambda^x}{x!}, x=1,2,....\text{and } k \text{ and } \lambda \text{ are positive constants.}$$

Find k in terms of λ.
Hence, find the mean value of X.

(ii) The lengths of rods produced by a certain machine are normally distributed with mean μ and variance 5.90 cm^2.
Given that, 5% of the rods are longer than 35 cm, find μ, giving your answer to 2 decimal places.
A rod is picked at random from amongst these rods.
Find the probability that it is shorter than 30 cm.
Five rods are picked at random. Find the probability that at least two of the rods are longer than 30 cm.
[Give all probabilities to 3 significant figures.]