

GCE ADVANCE LEVEL
FURTHER PURE MATHEMATICS
2010 PAPER 2

CGCEB – Further Mathematics Paper 2, JUNE 2010

1.
i. Solve the differential equation

$$(x + 2) \frac{dy}{dx} - y = (x + 2)^2,$$

given that when $x = 0$, $y = -4$.

- ii. Given that $y = Axe^x + Bxe^{2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^x + 2e^{2x},$$

find the value of the constants A and B .

Hence solve completely the differential equation given that when $y = 0$, $\frac{dy}{dx} = 1$ when $x = 0$.

2. The position vectors of the points **A**, **B**, **C**, **D** with respect to the origin **O** are **a**, **b**, **c**, **d** respectively, where
 $\mathbf{a} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Find:

- The Cartesian equation of the plane **ABC**.
- The Cartesian equation of the plane **BCD**.
- The cosine of the acute angle between the planes **ABC** and **BCD**.
- The area of the triangle **BCD**.
- The volume of the tetrahedron **ABCD**.

3. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ is $t^3x - ty = c(t^4 - 1)$.

The normal at P on the hyperbola meets the x -axis at Q and the tangent T meets the y -axis R . Show that the locus of the midpoint of QR , as P varies is $2c^2xy + y^4 = c^4$.

- 4.
- Find the root mean square value of $\tanh x$, for $0 \leq x \leq 2$.
 - A curve is given parametrically by $x = \cosh^2 t$, $y = \sinh^2 t$, $0 \leq t \leq 2$.
 - Find the length of the curve, leaving your answer in terms of e .
 - Prove that the area of surface generated by rotating the curve through 2π radians about the x -axis is given by $\frac{\pi}{3e^6}((e^4 + 1)^3 - 8e^6)$.

- 5.
- Prove that the set of numbers $\{1, 2, 4, 5, 7, 8\}$ forms an Abelian group under multiplication modulo 9.
 - Prove also that the set of numbers $\{1, 2, 4, 5\}$ forms an Abelian group under addition modulo 6.
- Are the two groups isomorphic? Give a reason to justify your answer.

- 6.
- Test each of the following series for convergence.

i. $\sum_{n=0}^{\infty} \left(\frac{2^n + 5}{3^n} \right)$

ii. $\sum_{n=1}^{\infty} \frac{n-1}{2n^2(n+1)}$

iii. $\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-3}$

- Find the first three terms of the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$.

Deduce that if $\left(x - \frac{\pi}{4}\right)$ is so small that $\left(x - \frac{\pi}{4}\right)^2$ and higher powers may be neglected, then $\tan x \approx 1 - \frac{\pi}{2} + 2x$.

7.

- a. Given that z_1 and z_2 are complex numbers, show geometrically, or otherwise that $|z_1| - |z_2| \leq |z_1 - z_2|$.
Hence or otherwise, show that if z is a complex number such that $|z^2 - 3z| = 4e^{i\alpha}$, where α is real, then $|z| \leq 4$.

- b. Given that $z = e^{i\theta}$, show that $z^n + \frac{1}{z^n} = 2 \cos n\theta$.

Hence, show that

$$\frac{\cos 5\theta}{\cos \theta} = 16 \sin^4 \theta - 112 \sin^2 \theta + 1,$$

for $\cos \theta \neq 0$.

- c. Show that the transformation $\omega = \frac{3z+6i}{iz-1}$ maps the line $|z+i| = |z+2i|$ to the curve $|\omega| = 3$.
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8.

- a. Solve for real x , in the equation $\sinh 2x - 2 \cosh 2x + 2 = 0$.
b. Express $\tanh x$ in terms of e^x and e^{-x} and hence show that

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

- c. Given that

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx,$$

show that $I_n + I_{n-2} = \frac{1}{n-1}$, ($n \geq 2$).

Hence, find $\int_0^{\frac{\pi}{8}} \tan^4 2x \, dx$.

9.

- a. Prove that if A and B are $n \times n$ non-singular matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.
b. Show that under the transformation represented by the matrix M , where

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & -2 & -7 \end{pmatrix},$$

the whole space is mapped onto the plane $x - 2y + z = 0$.

Find the image under this transformation of:

- i. The line $x = -y = \frac{z}{2}$,
ii. The plane $x - y + z = 0$, giving your answer in Cartesian form.
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