# GCE ADVANCE LEVEL

# FURTHER PURE MATHEMATICS

# 2011 PAPER 2

1. The points A, B, C, D have position vectors a, b, c, d, respectively, relative to an origin O, where $a = (3i - 2j + k)$, $b = (5i + j + k)$, $c = (2i + j + 4k)$, $d = (6i + 2j + k)$.

Find

(a) a Cartesian equation of the plane ABC,

(b) distance of the plane ABC from the origin,

(c) the volume of the tetrahedron ABCD,

(d) the equation of the plane $\Pi$ which passes through the point (-2, 0, 1) and is perpendicular to the plane ABC and $r.(i - 2j + 4k) = 6$

2. (a) Find the value of the constant $\lambda$ for which $\lambda x e^{2x}$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 10e^{2x}.$$

Obtain the general solution of the differential equation.

Find the solution for which $y = 0$ *and* $\frac{dy}{dx} = 5$, *when* $x = 0$.

(b) Solve the differential equation $x\frac{dy}{dx} + 2y = 4x^2$, giving that y = 2, when x = 1

3. (a) Find the complex number z which satisfy the equation $z^3 = 8i$, giving your answer in the form a + bi, where a and b are real.

(c) P is the point representing the complex number $z = r(\cos\theta + i\sin\theta)$ in an Argand diagram such that $|z - a||z + a| = a^2$. Show that P moves on the curve, whose equation is $r^2 = 2a^2\cos 2\theta$. Sketch the curve $r^2 = 2a^2\cos 2\theta$, showing clearly the tangents at the pole.

4. A linear transformation T is represented by the matrix $M$, where

$$M = \begin{pmatrix} 9 & -10 & 7 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}.$$

(a) Find the matrix product

$$\begin{pmatrix} 9 & -10 & 7 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \\ -1 & -1 & 5 \end{pmatrix}.$$

Deduce $M^{-1}$. Find, also,

(b) the image of the point (-2, -1, 3) under the Transformation T,

(c) the point, where the three planes

$2x + 3y - 5z = 2$

$x + 2y - z = 1$

$-x - y + 5z = 0$

intersect

(d) the Cartesian equation of the line, whose image is the line $\frac{x-1}{2} = y = z + 3$ under T.

5. (a) Find the real constants P, Q, R, for which

$\frac{3}{x^3+1} = \frac{P}{x+1} + \frac{Qx+R}{x^2-x+1}$. Show that $\int_0^1 \frac{3}{x^3+1}\,dx = \ln 2 + \frac{\pi\sqrt{3}}{3}$

(b) Given that $I_n = \int_0^{\frac{\pi}{3}} \sec^n x\,dx$, show that $(n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}, n \geq 2$

Evaluate $\int_0^{\frac{\pi}{3}} \sec^4 x\,dx$

6. (a) The line $5y - 3x + 1 = 0$ meets the hyperbola $x^2 - 3y^2 = 1$ at P and at Q.
The tangents at P and Q meet at R. Find,
(i) the coordinates of P, Q and R,
(ii) the area of triangle PQR

(b) A curve is given by the parametric equations $x = at,\ y = a(1-t)^{\frac{1}{2}}$ for $0 \leq t \leq 0$. Find the root mean square value of y with respect to x.

7. (a) Find the real values of x for which $3\sin^2 x + \cosh x = 1$

(b) Using the definitions of $\sinh x$ and $\cosh x$ in terms of $e^x$, show that $\coth x = 1 + \frac{2}{e^{2x}-1}$.

If $y = 1 + \frac{2}{e^{2x}-1}$, show that $e^{2x} = \frac{y+1}{y-1}$. Hence, or, otherwise, show that for real x, $\coth x$ cannot take values between − 1 and 1. Sketch the curve $y = \coth x$.

Evaluate $\int_{\ln 2}^{2\ln 2} \left(1 + \frac{2}{e^{2x}-1}\right)dx$

8. (i) Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. Prove by mathematical induction that, for all positive integers n,

$A^n = \begin{pmatrix} 1 & 0 \\ 1-2^n & 2^n \end{pmatrix}$.

Show that this formula, for $A^n$, is valid when $n = -1$

(ii) Consider the two groups

$G = (\Re, +)$ the set of real numbers under the usual addition and $H = (\Re^+, \times)$, the set of non negative real numbers under usual multiplication. Define a map f by

$f : \Re \rightarrow \Re^+$

$x \mapsto e^x$

Show that f defines an isomorphism from G to H

9. (a) A function f is defined by

$$f(x) = \begin{cases} kx + 1, & x < 1 \\ 3, & x = 1 \\ cx^2 + 2, & x > 1 \end{cases}$$

Find the values of c and k for which f is continuous over $0 \leq x \leq 4$

(b) Use the ratio test to determine the convergence or non convergence of the following series.

(i) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

(ii) $\sum_{n=1}^{\infty} \frac{3^{2n-1}}{n^2+n}$

(iii) $\sum_{n=1}^{\infty} \frac{n}{2^{2n}}$