# GCE ADVANCE LEVEL

# FURTHER PURE MATHEMATICS

# 2014 PAPER 2

1. (a) Using the substitution $z = y^2$, obtain the general solution of the differential equation

$$2xy\frac{dy}{dx} + y^2 = x$$

(b) Find the values of the constants a, b and c for which $y = ax^2 + bx + c$ is a particular integral to the differential equation $\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 4y = 8x^2 + 3$

Hence, slove completely the differential equation given that $y = 0$ *and* $\frac{dy}{dx} = 0$ *when* $x = 0$.

2. (a) Express $f(x) = \frac{4x^3 - x^2 + 5x - 2}{(x^2+2)(x^2+1)}$ into partial fractions.

Hence, show that $\int_0^1 f(x)dx = \frac{1}{2}\ln\left(\frac{27}{4}\right) - \frac{\pi}{4}$

(b) Given that $I_n = \int_1^e \left(\ln\frac{1}{x}\right)^n dx, n \geq 0$

Prove that $I_n = (-1)^n e + nI_{n-1}, n \geq 1$.

Hence, evaluate $\int_1^e \left(\ln\frac{1}{x}\right)^3 dx, n \geq 0$

3. (a) Test for the convergence of each of the following series

(i) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

(ii) $\sum_{n=1}^{\infty} \frac{2n+5}{n^2+3n+2}$

(iii) $\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}$

(b) Given that the first two non-zero terms in the maclaurin expansion of $e^{ax}\cos(2x) - \ln(1+bx) - 1$ are $7x^2$ *and* $\frac{21}{2}x^3$, find the values of $a$ *and* $b$

4. (a) Given the function $f$, where $f(x) = x|x| - x, x \in \mathbb{R}$

(i) Sketch the curve $y = f(x)$

(ii) Hence, or otherwise, solve the inequality $x|x| - x \geq x$

(iii) Show that the curve $y = f(x)$ is invariant under the ...

$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Find

(i) the eccentricity

(ii) the coordinates of the foci

(iii) the equation of the asymptotes

**5.** (a) Express $z = \frac{1}{2}\left(1 + i\sqrt{3}\right)$ in the form $r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$

Using de Moivre's theorem show that $Z$ is a root of the equation $z^4 + z^2 + 1 = 0$

Hence, or otherwise, obtain the other roots of the equation in the form $a + ib$ where $a$ and $b$ are real

Indicate on an Argand diagram, the points A, B, C, D representing these roots

Find the area of the rectangle ABCD.

(b) A transformation T from the z-plane to the w-plane is given by $\omega = \frac{z+1}{z-1}, z \neq 1$

Find the image in the w-plane of the circle $|z| = 1, z \neq 1$, under the transformation T and interpret your result.

**6.** (a) A plane passes through three points P, Q and R whose position vectors are $2i - j + k, 3i + 2j - 2k$ and $-i + 3j + 2k$ respectively.

Show that the Cartesian equation of the plane PQR is $11x + 5y + 13z = 30$

(b) Show that the transformation with matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ maps the whole space onto the plane $2x - y = 0$

(i) Show that the line $x = \frac{y}{2} = z - 1$

Lies on the plane $2x - y = 0$ and find its image under the transformation.

(ii) Find a vector equation of the line whose image under the transformation is the point $(1, 2, 4)$

**7.** (a) Let G be a group defined by multiplication on the set $\{x, x^2, x^3, x^4\}, x \in \mathbb{R}$ such that $x^5 = x$

Draw the group table for G

Find the identity element and show that G is a commutative group

(b) Show that the set $H = \{1, 3, 7, 9\}$ forms a group under multiplication modulo 10.

(c) Show further that the groups in (a) and (b) are isomorphic.

**8.** (a) Prove by induction that

$$\sum_{r=1}^{n} 4^r = \frac{4}{3}(4^n - 1),$$ where $n$ is a positive integer

(b) Given the real function $f$, where

(i) Find $\lim_{x \to 1} f(x)$

(ii) Sketch the curve $y = f(x)$.

(iii) Study the continuity of the function. $f$

**9.** (a) Solve for real $x$, the equation $3\sinh^2 x - 2\cosh x + 2 = 0$

(b) Prove that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ Hence show that $\int_0^1 \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \ln\left(\frac{2 + \sqrt{5}}{1 + \sqrt{2}}\right)$