GENERAL CERTIFICATE OF EDUCATION BOARD

General Certificate of Education Examination

FURTHER MATHEMATICS PAPER 2
0775

JUNE 2022 **ADVANCED LEVEL**

Subject Title	**Further Mathematics**
Paper No.	**Paper 2**
Subject Code No.	**0775**

THREE hours

Answer ALL 10 questions.

For your guidance, the approximate mark allocation for parts of each question is indicated.

Mathematical formulae and tables published by the Board, and noiseless non-programmable electronic calculators are allowed.

In calculations, you are advised to show all the steps in your working, giving your answer at each stage.

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1. Find the complementary function of the differential equation

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=e^{-3x},$$

in the form $y=f(x)$. **(3 marks)**

Hence find the particular integral and the general solution in the form $y=f(x)$. **(4 marks)**

2. Express $f(x)$ in partial fractions, where

$$f(x)=\frac{2x^3+x+2}{(x^2+1)(x^2-x+1)}.$$ **(4 marks)**

Hence, or otherwise, show that

$$\int_0^1 f(x)\,dx=\ln 2+\frac{\pi}{36}\left(9+8\sqrt{3}\right).$$ **(4 marks)**

3. a) Solve the equation

$$2\tanh^{-1}\left(\frac{x-2}{x+1}\right)=\ln 2.$$ **(4 marks)**

b) Show that the set $\{1,4,7,13\}$ under $\times_{15}$, multiplication mod 15, forms a group. **(4 marks)**

4. (a) A sequence, (u_n), is defined recursively by

$$u_1=1 \text{ and } u_{n+1}=3u_n-1, \quad \forall n\in\mathbb{N}.$$

Prove by induction that

$$u_n=\frac{1}{2}(1+3^{n-1}).$$ **(4 marks)**

b) A curve is given by the parametric equations

$$x=2\theta-\sin 2\theta,\ y=\cos 2\theta.$$

Show that the length of the curve for $\frac{\pi}{6}\le\theta\le\frac{\pi}{3}$ is $2\left(\sqrt{3}-1\right)$ **(3 marks)**

5. Show that the curve with polar coordinates (r,θ), where

$$1-\cos\theta=\frac{2(r-3)}{r(r+2)},\quad r\ge 3$$

is a parabola, P, in the (x,y) plane. **(2 marks)**

Show also that the point $(11,4)$ lies on P and find the equation of the tangent to P at this point. **(4 marks)**

6. a) By the use of the Chinese Remainder Theorem, or otherwise, solve the system of congruences

$$x\equiv 2 \pmod 5$$
$$x\equiv 5 \pmod 7$$ **(5 marks)**

b) A complex number z is defined by $z=\frac{1}{2}(\cos\theta+i\sin\theta)$, such that

$$z^n=\frac{1}{2^n}(\cos n\theta+i\sin n\theta),\ n\in\mathbb{N}.$$

Show that

i) $\sum_{r=0}^{\infty}\frac{1}{4^r}\cos 2r\theta$ is a convergent geometric progression. **(1 mark)**

ii) $\sum_{r=0}^{\infty} \frac{1}{4^r} \cos 2r\theta = \frac{4(4-\cos 2\theta)}{17-16\cos 2\theta}$ **(4 marks)**

7. A transformation, f, on a complex plane is defined by

$$z' = -iz + 2i\ .$$

(i) Find the image of the point

$$z = \frac{3}{2} + i\ .$$

(1 mark)

(ii) Determine the invariant point of f in the form $a + ib,\ \ a, b \in \mathbb{R}$. **(2 marks)**

(iii) Show that f is a similarity transformation (similitude), stating its radius. **(2 marks)**

(iv) Give the geometrical interpretation of the transformation f. **(1 mark)**

8. Given two vectors

$$\mathbf{a} = (2-\lambda)\mathbf{i} - \mathbf{j} - 4\mathbf{k} \quad \text{and}$$

$$\mathbf{b} = (3+\lambda)\mathbf{i} + 2\mathbf{j} + (3+\mu)\mathbf{k},\ \lambda, \mu \in \mathbb{Z}$$

such that

$$\mathbf{a} \times \mathbf{b} = 4\mathbf{i} - 20\mathbf{j} + 6\mathbf{k},$$

i. Calculate the values of the real constants λ and μ. **(3 marks)**

ii. By using the values of λ and μ, state the vectors $\mathbf{a}$ and $\mathbf{b}$. **(1 mark)**

iii. Show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. **(2 marks)**

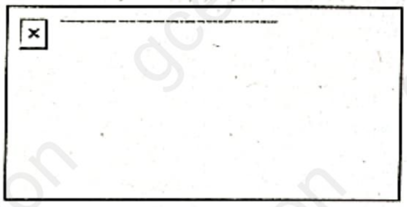
iv. Find the Cartesian equation of the plane containing $\mathbf{a}$ and $\mathbf{b}$. **(2 marks)**

9. Given that

$$f(x) = \sin x$$

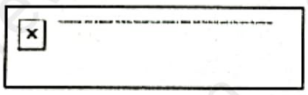
i. State the range and period of f. **(3 marks)**

Another function g is defined by



Show that

ii.  **(2 marks)**

iii. $-\frac{1}{x} \le g(x) \le \frac{1}{x},\ x > 0$ and . **(4 marks)**

iv. $\lim_{x\to\infty} \frac{\sin x}{x} = 0.$ **(2 marks)**

v. $g(x)$ is an even function. **(2 marks)**

vi. On the same coordinates axes sketch the curves

$$y = -\frac{1}{x},\ y = \frac{1}{x} \quad \text{and}$$ **(3 marks)**

$$y = g(x)$$ **(4 marks)**

Turn Over

10. A sequence (u_n) is defined recursively by

$$u_0 = \frac{1}{2}, \quad u_{n+1} = \frac{e^{u_n}}{u_n + 2}$$

Given that a function, $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \frac{e^x}{x+2},$$

i) Find $f'(x)$ and $f''(x)$. **(3 marks)**

ii) Obtain a variation table of f. **(3 marks)**

iii) Use Mathematical induction to show that for all

$$n \in \mathbb{N}, 0 \leq u_n \leq 1.$$ **(3 marks)**

Show that

iv) for all $x \in [0,1]$, $|f'(x)| \leq \frac{2}{3}$. **(3 marks)**

v) the equation $f(x) = x$ has a unique solution. **(2 marks)**

vi) for $l \in [0,1]$, $|u_{n+1} - l| \leq \frac{2}{3}|u_n - l|$ for all $n \in \mathbb{N}$ and that in general,

$$|u_n - l| \leq \frac{1}{2}\left(\frac{2}{3}\right)^n.$$ **(3 marks)**

vii) Hence show, also, that the sequence (u_n) is convergent. **(3 marks)**

GO BACK AND CHECK YOUR WORK