FURTHER MATHEMATICS 2

0775/2

WEST MATHEMATICS TEACHERS' PEDAGOGIC GROUP

GENERAL CERTIFICATE OF EDUCATION MOCK EXAMINATION

24th MARCH, 2023 **ADVANCED LEVEL**

Subject Title	**Further Mathematics**
Paper No.	**Paper 2**
Subject Code No.	**0775**

THREE HOURS

Answer ALL 10 questions.

For your guidance, the approximate mark allocation for parts of each question is indicated.

Mathematical formulae booklets and tables published by the GCE Board, and noiseless non-programmable electronic calculators are allowed.

In calculations, you are advised to show all the steps in your working, giving the answer at each stage.

Start each question on a fresh page

Turn Over

1. Find the complementary function of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^{2x}$ in the form $y = f(x)$. **(3 marks)**

Hence, find the particular integral and general solution in the form $y = f(x)$.

(4 marks)

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2. Express $f(x)$ in partial fractions, where $f(x) = \frac{5x^3 + x + 1}{(x^2 - x + 1)(x^2 + 1)}$ **(4 marks)**

Hence, or otherwise, show that $\int_0^1 f(x)\,dx = \pi + \frac{1}{2}\ln 2 - \frac{2\pi}{3\sqrt{3}}$

(5 marks)

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3. (i) Solve the equation; $\sinh^2 x - 2\cosh x + 2 = 0$. **(3 marks)**

(ii) Use the definition of $\cosh x$ in terms of exponential function, to prove that

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x^2 > 1.$$

(3 marks)

(iii) Functions f, g, h and k are defined for $x \neq 0$, as follows.

$$f : x \mapsto x \qquad g : x \mapsto \frac{1}{x} \qquad h : x \mapsto -x \qquad k : x \mapsto -\frac{1}{x}$$

Show that the set $\{f, g, h, k\}$ under composition of functions forms a group. You may assume that the associative axiom is satisfied.

(5 marks)

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4. (i) Show that the curve with polar coordinates (r, θ), where $r = \frac{32}{3 + 5\sin\theta}$, is a hyperbola, P, in the $xy-$plane; and give the equations of the asymptotes to the hyperbola.

(6 marks)

(ii) Prove by contradiction that $\frac{1}{\sqrt{7}}$ is irrational.

(3 marks)

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5. (i) A plane curve has parametric equations $x = 2\cos 2\theta, \quad y = 2\sin 2\theta$. Find the length of the curve in the interval $0 \leq \theta \leq 2\pi$. **(3 marks)**

(ii) The curve with polar coordinates (r, θ) is defined by $r = 1 + 2\sin\theta$.

(a) Show that the line $\theta = \frac{\pi}{2}$ is a tangent line parallel to the initial line. **(3 marks)**

(b) Find the tangent line at the pole. **(3 marks)**

(c) Find the coordinates of the point that is furthest form the pole.

(1 mark)

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6. (i) By use of the Chinese Remainder Theorem, or otherwise, solve the system of congruence;

$$\begin{aligned} x &\equiv 2 \pmod 3) \\ x &\equiv 5 \pmod 7) \\ x &\equiv 8 \pmod{11}) \end{aligned}$$

(6 marks)

(ii) A complex number z is defined by $z = e^{i\theta} = \left(\cos\theta + i\sin\theta\right)$,. Show that

(a) $\sum_{r=1}^{n} z^{2r-1} = \frac{z - z^{2n+1}}{1 - z^2}$ and $\overline{1 - z^2} = 1 - \bar{z}^2$. **(3 marks)**

(b) Hence or otherwise, show that $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{1 - \cos 2n\theta}{2\sin\theta}$

(4 marks)

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7. A transformation f on the complex plane is defined as $z' = (1+i)z + 2i$.

(a) Find the image of the point $z = 1 - 2i$. **(1 mark)**

(b) Determine the invariant point of f in the form $a + ib$. **(2 marks)**

(c) Show that f is a similarity transformation (similitude), stating its radius. **(3 marks)**

(d) Give the geometrical interpretation of the transformation f.

(1 mark)

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8. Given two vectors $\mathbf{a} = (\lambda-1)\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = (\lambda-2)\mathbf{i} - 2\mathbf{j} + (\mu+3)\mathbf{k}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, with $\lambda, \mu \in \mathbb{Z}$.

(a) Calculate the values of the real constants λ and μ. **(3 marks)**

(b) By using these values, state the vectors $\mathbf{a}$ and $\mathbf{b}$. **(1 mark)**

(c) Show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. **(2 marks)**

(d) Find the Cartesian equation of the plane containing $\mathbf{a}$ and $\mathbf{b}$.

(2 marks)

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9. A numerical function f of real variable x is defined by

$$f(x) = 2x + 3 + \ln\left(\frac{x+1}{x-1}\right).$$

(a) Determine the domain of definition, D_f of f. **(2 marks)**

(b) Calculate the limits at the bounds of D_f. **(3 marks)**

(c) Evaluate $\lim_{x\to\pm\infty} \left[f(x) - (2x+3)\right]$.
Hence, state all the asymptotes to the curve C_f of f.

(3 marks)

(d) Show that $f'(x) = \frac{2x^2 - 4}{x^2 - 1}$ and construct the table of variation of f. **(3 marks)**

(e) Sketch the graph of $y = f(x)$ showing clearly all asymptotes and the behavior of the curve as it approaches the asymptotes. **(2 marks)**

(f) Hence, find the coordinates of the centre of symmetry of the curve.

(1 mark)

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10. (i) The sequences u_n and v_n are defined by

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \frac{1}{4}(3u_n + 1) \end{cases} \text{ and } \begin{cases} v_0 = 2 \\ v_{n+1} = \frac{1}{4}(3v_n + 1) \end{cases}$$

Let $w_n = v_n - u_n$ and $t_n = v_n + u_n$.

(a) Show that (w_n) is a geometric sequence and give its nth term in terms of n.

(3 marks)

(b) Find $\lim_{n \to +\infty} (w_n)$ **(1 mark)**

(c) Prove by induction that (t_n) is a constant sequence. **(3 marks)**

(d) Hence, write (u_n) and (v_n) in terms of n. **(2 marks)**

(e) Show that (u_n) and (v_n) are convergent.

(2 marks)

(ii) The sequence (x_n) is defined by

$$x_n x_{n-1} - 4x_n - 8x_{n-1} + 36 = 0.$$

If $\lim_{x \to +\infty} (x_n) = l$. Find the possible values of l.

(2 marks)
