GENERAL CERTIFICATE OF EDUCATION BOARD

General Certificate of Education Examination

FURTHER MATHEMATICS PAPER 2
0775

JUNE 2023 **ADVANCED LEVEL**

Subject Title	**Further Mathematics**
Paper No.	**Paper 2**
Subject Code No.	**0775**

THREE HOURS

Answer ALL 10 questions.

For your guidance, the approximate mark allocation for parts of each question is indicated.

Mathematical formulae and tables published by the Board, and noiseless non-programmeable electronic calculators are allowed.

In calculations, you are advised to show all the steps in your working, giving your answer at each stage.

1. a) Use the substitution $xy = v$, where v is a function of x, to transform the differential equation

$$x^2 \frac{dy}{dx} + xy = 2\left(x^2y^2 + 1\right).$$ **(2 marks)**

into a differential equation involving v and x only.

Hence, find the general solution of this differential equation in terms of x and y. **(3 marks)**

b) Given that $A\cos 4x + B\sin 4x$ is a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 14\sin 4x.$$

Find the values of the real constants A and B. **(2 marks)**

Hence, find the general solution of the differential equation. **(3 marks)**

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2. Given two vectors

$$\mathbf{a} = x\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{b} = \mathbf{i} + y\mathbf{j} - \mathbf{k}, \quad x, y \in \mathbb{Z},$$

and that $\mathbf{a} \times \mathbf{b} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

i) Calculate the values of the real constants, x and y. **(3 marks)**

ii) Show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. **(2 marks)**

iii) Find the Cartesian equation of the plane that contains $\mathbf{a}$ and $\mathbf{b}$ and passes through the point with position vector $\mathbf{j} + \mathbf{k}$. **(2 marks)**

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3. Prove that

$$\tanh^{-1} 4x = \tfrac{1}{2}\ln\left(\frac{1+4x}{1-4x}\right), \ |x| < \tfrac{1}{4}.$$ **(3 marks)**

Hence, or otherwise, solve the equation $\tanh^{-1} 4x - \ln 2 = 0$. **(3 marks)**

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4. Given that

$$f(x) = \frac{3x^2 + 10x + 9}{(x+2)^4}, x \neq -2,$$

Use the substitution $u = x + 2$, or otherwise, to express $f(x)$ in partial fractions. **(4 marks)**

Hence, show that

$$\int_{-1}^{0} f(x)\,dx = \frac{25}{24}.$$ **(4 marks)**

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5. a) Show that the set of matrices of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \text{ where } x \neq 0, \text{ and } x, y \in \mathbb{R},$$

forms a group under matrix multiplication. (*Assume Associativity*). **(5 marks)**

b) Solve the linear congruence

$$x - 3 \equiv 5 (\bmod 7),$$

giving your answer in the form $x = p\mu + q$, where $p, q \in \mathbb{N}$ and $\mu \in \mathbb{Z}$. **(3 marks)**

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6. Show that the curve with polar coordinates (r, θ) where,

$$r^2 = \frac{36}{9 - 5\sin^2\theta},$$

represents an ellipse. **(3 marks)**

Find in polar form, the coordinates of the foci of this ellipse. **(5 marks)**

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7. a) Solve completely the complex equation

$$z^3 = -8i.$$

(4 marks)

b) Find the centre and scale factor of the transformation described by

$$w = 2z - 3 + 4i.$$

(2 marks)

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8. a) The parametric equations of a curve C are given by

$$x = \cos t - 1 \text{ and } y = \sin t, \text{ for } 0 < t < \frac{\pi}{3},$$

Find, L, the length of C. **(5 marks)**

b) A transformation T is defined by matrix M, where

$$\mathrm{M} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

i) Show that T is an isometry. **(1 mark)**

ii) Find the invariant point under T. **(2 marks)**

iii) Show that the image of the line

$L : x = 2y$ is the line $L' : 7x = 10y$. **(2 marks)**

iv) Hence, or otherwise, find the angle of rotation under T. **(1 mark)**

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9. a) A recursive sequence, is defined by

$$u_1 = 3, \qquad u_2 = 13$$

$$u_{n+1} = 3u_n + 4u_{n-1} + 2^n, n \in \mathbb{N}.$$

i) Show that

$$u_{n+1} + u_n + 2^n = 4(u_n + u_{n-1} + 2^{n-1}).$$

(1 mark)

Another sequence v_n is defined by $v_n = u_n + u_{n-1} + 2^{n-1}$.

ii) Show further that (v_n) is a geometric sequence. **(2 marks)**

iii) Show that $v_1 = \frac{9}{2}$ **(3 marks)**

Turn Over

iv) Find (v_n) in terms of n. **(3 marks)**

Given also that

$$u_{n+1} - 4u_n - \tfrac{2}{3}(2^n) = -\left(u_n - 4u_{n-1} - \tfrac{2}{3}(2^{n-1})\right).$$

v) Show that

$$u_{n+1} - 4u_n = \tfrac{1}{3}\left(2^n - (-1)^n\right).$$

(3 marks)

vi) Hence, or otherwise, deduce that

$$5u_n = \tfrac{9}{2}(4^n) - \tfrac{1}{3}(-1)^n - \tfrac{4}{3}(2^n).$$ **(2 marks)**

b) A sequence is defined by

$$u_n = \frac{1}{2\left(3^n + 1\right)},\ n \geq 1.$$

Show that

$$\sum_{n=1}^{\infty} u_n < \frac{1}{4}.$$ **(4 marks)**

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10. A function f, is defined by,

$$f(x) = \frac{x^3}{\left(x+1\right)^2}.$$

i) State the domain of f. **(1 mark)**

ii) State the coordinates of the intercepts of $y = f(x)$. **(1 mark)**

iii) Find the asymptotes to the curve $y = f\left(x\right)$. **(3 marks)**

iv) Find the stationary point of the curve $y = f\left(x\right)$. **(3 marks)**

v) Find the intervals in which f is increasing and in which it is decreasing. **(3 marks)**

vi) Find the point of inflexion of $y = f\left(x\right)$. **(2 marks)**

vii) Draw a variation table for f. **(2 marks)**

viii) Sketch the curve of $y = f\left(x\right)$. **(2 marks)**

GO BACK AND CHECK YOUR WORK.