NWRM 2025 0775 P2 Solutions

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Contents

1 Second Order Differential Equation

Problem Statement

Consider the second order differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin(x)$

a) Find the complementary function (2 marks)

b) Find a particular solution to the non-homogeneous differential equation (3 marks)

c) Find the general solution of the differential equation given that $y(0) = 0$ and $y'(0) = 1$ (3 marks)

Part (a): Finding the Complementary Function

Solution: The complementary function is the general solution to the corresponding homogeneous differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \tag{1}$$

To solve this, we use the auxiliary equation method. Let's assume a solution of the form $y = e^{rx}$. Substituting this into the homogeneous equation:

$$r^2e^{rx} + re^{rx} = 0 \tag{2}$$

$$e^{rx}(r^2 + r) = 0 \tag{3}$$

Since $e^{rx} \neq 0$ for any finite value of x, we have:

$$r^2 + r = 0 \tag{4}$$

$$r(r + 1) = 0 \tag{5}$$

This gives us two roots:

$$r_1 = 0 \tag{6}$$

$$r_2 = -1 \tag{7}$$

Therefore, the complementary function is:

$$y_c(x) = C_1 + C_2 e^{-x} \tag{8}$$

where C_1 and C_2 are arbitrary constants.

Part (b): Finding a Particular Solution

Solution: For the non-homogeneous term $\sin(x)$, we'll use the method of undetermined coefficients. Since $\sin(x)$ appears on the right-hand side, we try a particular solution of the form:

$$y_p(x) = A\sin(x) + B\cos(x) \tag{9}$$

Taking derivatives:

$$y_p'(x) = A\cos(x) - B\sin(x) \tag{10}$$

$$y_p''(x) = -A\sin(x) - B\cos(x) \tag{11}$$

Substituting into the original differential equation:

$$-A\sin(x) - B\cos(x) + A\cos(x) - B\sin(x) = \sin(x) \tag{12}$$

$$(-A - B)\sin(x) + (A - B)\cos(x) = \sin(x) \tag{13}$$

Comparing coefficients:

$$-A - B = 1 \quad \text{(coefficient of } \sin(x)) \tag{14}$$

$$A - B = 0 \quad \text{(coefficient of } \cos(x)) \tag{15}$$

From the second equation, we get $A = B$. Substituting into the first equation:

$$-A - A = 1 \tag{16}$$

$$-2A = 1 \tag{17}$$

$$A = -\frac{1}{2} \tag{18}$$

Therefore, $A = B = -\frac{1}{2}$, and our particular solution is:

$$y_p(x) = -\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) \tag{19}$$

Part (c): Finding the General Solution with Initial Conditions

Solution: The general solution is the sum of the complementary function and the particular solution:

$$y(x) = y_c(x) + y_p(x) = C_1 + C_2e^{-x} - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) \tag{20}$$

We need to find the values of C_1 and C_2 using the initial conditions:

$$y(0) = 0 \tag{21}$$

$$y'(0) = 1 \tag{22}$$

First, let's find $y'(x)$:

$$y'(x) = -C_2e^{-x} - \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) \tag{23}$$

Using the first initial condition, $y(0) = 0$:

$$y(0) = C_1 + C_2 - \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0) \tag{24}$$

$$0 = C_1 + C_2 - \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0 \tag{25}$$

$$0 = C_1 + C_2 - \frac{1}{2} \tag{26}$$

$$C_1 + C_2 = \frac{1}{2} \tag{27}$$

Using the second initial condition, $y'(0) = 1$:

$$y'(0) = -C_2 - \frac{1}{2}\cos(0) + \frac{1}{2}\sin(0) \tag{28}$$

$$1 = -C_2 - \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 \tag{29}$$

$$1 = -C_2 - \frac{1}{2} \tag{30}$$

$$1 + \frac{1}{2} = -C_2 \tag{31}$$

$$\frac{3}{2} = -C_2 \tag{32}$$

$$C_2 = -\frac{3}{2} \tag{33}$$

Substituting this value of C_2 into the equation $C_1 + C_2 = \frac{1}{2}$:

$$C_1 + \left(-\frac{3}{2}\right) = \frac{1}{2} \tag{34}$$

$$C_1 = \frac{1}{2} + \frac{3}{2} \tag{35}$$

$$C_1 = 2 \tag{36}$$

Therefore, the general solution with the given initial conditions is:

$$y(x) = 2 - \frac{3}{2}e^{-x} - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x) \tag{37}$$

Answer: Part (a): The complementary function is $y_c(x) = C_1 + C_2e^{-x}$
Part (b): A particular solution is $y_p(x) = -\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)$
Part (c): The general solution satisfying the initial conditions is: $y(x) = 2 - \frac{3}{2}e^{-x} - \frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)$

2 Partial Fractions and Integration

Problem Statement

a) Express $\frac{x^2-3}{(x+1)^2(x^2+1)}$ in partial fractions. (4 marks)

b) Prove that the reduction formula for the integral $I_n = \int_0^1 x^n e^{Ax} dx$ is given by $AI_n = e^A - nI_{n-1}$ where $n \geq 1$. Hence, evaluate $\int_0^1 x^3 e^{2x} dx$ (5 marks)

Part (a): Partial Fractions Decomposition

Solution: For the rational function $\frac{x^2-3}{(x+1)^2(x^2+1)}$, we seek a partial fractions decomposition of the form:

$$\frac{x^2-3}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)} \tag{38}$$

Multiplying both sides by $(x+1)^2(x^2+1)$:

$$x^2 - 3 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \tag{39}$$

$$x^2 - 3 = A(x^3+x^2+x+1) + B(x^2+1) + (Cx+D)(x^2+2x+1) \tag{40}$$

$$x^2 - 3 = A(x^3+x^2+x+1) + B(x^2+1) + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D \tag{41}$$

Expanding and collecting terms:

$$x^2 - 3 = Ax^3 + Ax^2 + Ax + A + Bx^2 + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D \tag{42}$$

$$x^2 - 3 = (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D) \tag{43}$$

Comparing coefficients of powers of x:

$$x^3 : A + C = 0 \quad \Rightarrow \quad C = -A \tag{44}$$

$$x^2 : A + B + 2C + D = 1 \tag{45}$$

$$x^1 : A + C + 2D = 0 \tag{46}$$

$$x^0 : A + B + D = -3 \tag{47}$$

Let's solve this system of equations. From $A + C = 0$ and $C = -A$, substituting into the third equation:

$$A + (-A) + 2D = 0 \quad (48)$$
$$2D = 0 \quad (49)$$
$$D = 0 \quad (50)$$

From the fourth equation with $D = 0$:

$$A + B + 0 = -3 \quad (51)$$
$$A + B = -3 \quad (52)$$

Now, using the second equation with $C = -A$ and $D = 0$:

$$A + B + 2(-A) + 0 = 1 \quad (53)$$
$$A + B - 2A = 1 \quad (54)$$
$$-A + B = 1 \quad (55)$$

We now have two equations:

$$A + B = -3 \quad \ldots(1) \quad (56)$$
$$-A + B = 1 \quad \ldots(2) \quad (57)$$

Adding equations (1) and (2):

$$2B = -2 \quad (58)$$
$$B = -1 \quad (59)$$

Substituting back into equation (1):

$$A + (-1) = -3 \quad (60)$$
$$A = -3 + 1 \quad (61)$$
$$A = -2 \quad (62)$$

With $A = -2$, we can confirm $C = -A = -(-2) = 2$.
Therefore, our partial fractions decomposition is:

$$\frac{x^2 - 3}{(x+1)^2(x^2+1)} = \frac{-2}{(x+1)} + \frac{-1}{(x+1)^2} + \frac{2x}{(x^2+1)} \quad (63)$$

Let's verify our solution by checking the coefficients in the expanded numerator:

$$x^3 : -2 + 2 = 0\checkmark \quad (64)$$
$$x^2 : -2 - 1 + 2(2) + 0 = -2 - 1 + 4 = 1\checkmark \quad (65)$$
$$x^1 : -2 + 2 + 2(0) = 0\checkmark \quad (66)$$
$$x^0 : -2 - 1 + 0 = -3\checkmark \quad (67)$$

Our decomposition is verified.

Part (b): Reduction Formula and Integral Evaluation

Solution: We need to prove the reduction formula $AI_n = e^A - nI_{n-1}$ where $I_n = \int_0^1 x^n e^{Ax} dx$ and $n \geq 1$.

Let's use integration by parts with:

$$u = x^n \quad \Rightarrow \quad du = nx^{n-1}dx \quad (68)$$
$$dv = e^{Ax}dx \quad \Rightarrow \quad v = \frac{e^{Ax}}{A} \quad (69)$$

Applying the formula $\int u\,dv = uv - \int v\,du$:

$$I_n = \int_0^1 x^n e^{Ax} dx \quad (70)$$
$$= \left[x^n \cdot \frac{e^{Ax}}{A}\right]_0^1 - \int_0^1 \frac{e^{Ax}}{A} \cdot nx^{n-1}dx \quad (71)$$
$$= \frac{1^n \cdot e^{A\cdot 1}}{A} - \frac{0^n \cdot e^{A\cdot 0}}{A} - \frac{n}{A}\int_0^1 x^{n-1}e^{Ax}dx \quad (72)$$
$$= \frac{e^A}{A} - \frac{n}{A} \cdot I_{n-1} \quad (73)$$

Multiplying both sides by A:

$$AI_n = e^A - nI_{n-1} \quad (74)$$

Which is the required reduction formula.

Now, to evaluate $\int_0^1 x^3 e^{2x} dx$, we need to use the reduction formula with $A = 2$ and $n = 3$.

First, let's define $I_n = \int_0^1 x^n e^{2x} dx$. Then:

$$2I_3 = e^2 - 3I_2 \tag{75}$$
$$\Rightarrow I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 \tag{76}$$

Similarly:

$$2I_2 = e^2 - 2I_1 \tag{77}$$
$$\Rightarrow I_2 = \frac{e^2}{2} - I_1 \tag{78}$$

And:

$$2I_1 = e^2 - 1 \cdot I_0 \tag{79}$$
$$\Rightarrow I_1 = \frac{e^2}{2} - \frac{I_0}{2} \tag{80}$$

For I_0, we have:

$$I_0 = \int_0^1 e^{2x} dx \tag{81}$$
$$= \left[\frac{e^{2x}}{2}\right]_0^1 \tag{82}$$
$$= \frac{e^2 - 1}{2} \tag{83}$$

Now, we can work backwards:

$$I_1 = \frac{e^2}{2} - \frac{I_0}{2} \tag{84}$$
$$= \frac{e^2}{2} - \frac{1}{2} \cdot \frac{e^2 - 1}{2} \tag{85}$$
$$= \frac{e^2}{2} - \frac{e^2 - 1}{4} \tag{86}$$
$$= \frac{2e^2 - e^2 + 1}{4} \tag{87}$$
$$= \frac{e^2 + 1}{4} \tag{88}$$

Next:

$$I_2 = \frac{e^2}{2} - I_1 \quad (89)$$

$$= \frac{e^2}{2} - \frac{e^2+1}{4} \quad (90)$$

$$= \frac{2e^2 - e^2 - 1}{4} \quad (91)$$

$$= \frac{e^2 - 1}{4} \quad (92)$$

Finally:

$$I_3 = \frac{e^2}{2} - \frac{3}{2} I_2 \quad (93)$$

$$= \frac{e^2}{2} - \frac{3}{2} \cdot \frac{e^2-1}{4} \quad (94)$$

$$= \frac{e^2}{2} - \frac{3(e^2-1)}{8} \quad (95)$$

$$= \frac{4e^2 - 3e^2 + 3}{8} \quad (96)$$

$$= \frac{e^2+3}{8} \quad (97)$$

Therefore:

$$\int_0^1 x^3 e^{2x} dx = \frac{e^2+3}{8} \quad (98)$$

Answer: Part (a): The partial fractions decomposition is:

$$\frac{x^2-3}{(x+1)^2(x^2+1)} = \frac{-2}{(x+1)} + \frac{-1}{(x+1)^2} + \frac{2x}{(x^2+1)} \quad (99)$$

Part (b): The value of the integral $\int_0^1 x^3 e^{2x} dx = \frac{e^2+3}{8}$

3 Hyperbolic Functions and Group Homomorphisms

Problem Statement

a) Solve for x the equation $\cosh(x - \ln 2) = \frac{5}{4}$ (3 marks)

b) Let $G = (R, +)$ be the group of real numbers under addition. Let $H = (R \setminus \{0\}, \times)$ be the group of non-zero real numbers under multiplication. A mapping from G to H is defined by $f(x) = 2^x$. Prove that f is a homomorphism. Also determine whether or not f is an isomorphism. (5 marks)

Part (a): Solving the Hyperbolic Equation

Solution: We need to solve the equation $\cosh(x - \ln 2) = \frac{5}{4}$.

First, recall the definition of the hyperbolic cosine function:

$$\cosh(y) = \frac{e^y + e^{-y}}{2} \tag{100}$$

Let's substitute $y = x - \ln 2$ into the equation:

$$\cosh(y) = \frac{5}{4} \tag{101}$$

$$\frac{e^y + e^{-y}}{2} = \frac{5}{4} \tag{102}$$

$$e^y + e^{-y} = \frac{10}{4} \tag{103}$$

$$e^y + e^{-y} = \frac{5}{2} \tag{104}$$

Let's introduce a substitution $z = e^y$, which gives $e^{-y} = \frac{1}{z}$. Then our equation becomes:

$$z + \frac{1}{z} = \frac{5}{2} \tag{105}$$

Multiplying both sides by z:

$$z^2 + 1 = \frac{5z}{2} \tag{106}$$

$$z^2 - \frac{5z}{2} + 1 = 0 \tag{107}$$

This is a quadratic equation in z. Using the quadratic formula:

$$z = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} \tag{108}$$

$$= \frac{\frac{5}{2} \pm \sqrt{\frac{25-16}{4}}}{2} \tag{109}$$

$$= \frac{\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2} \tag{110}$$

$$= \frac{\frac{5}{2} \pm \frac{3}{2}}{2} \tag{111}$$

$$= \frac{5 \pm 3}{4} \tag{112}$$

So $z = 2$ or $z = \frac{1}{2}$.
Recall that $z = e^y$, so $y = \ln(z)$:

$$y = \ln(2) \quad \text{or} \quad y = \ln\left(\frac{1}{2}\right) \tag{113}$$

$$y = \ln(2) \quad \text{or} \quad y = -\ln(2) \tag{114}$$

Substituting back $y = x - \ln 2$:

$$x - \ln 2 = \ln 2 \quad \text{or} \quad x - \ln 2 = -\ln 2 \tag{115}$$

$$x = 2\ln 2 \quad \text{or} \quad x = 0 \tag{116}$$

Therefore, the solutions to the equation $\cosh(x - \ln 2) = \frac{5}{4}$ are $x = 2\ln 2$ or $x = 0$.

Part (b): Group Homomorphism and Isomorphism

Solution: We need to prove that $f : G \to H$ defined by $f(x) = 2^x$ is a homomorphism, where $G = (R, +)$ and $H = (R \setminus \{0\}, \times)$.

Proving that f is a homomorphism

For a function to be a homomorphism from $(G, +)$ to $(H, \times)$, it must preserve the group operation. Specifically, for all $x, y \in R$, we must have:

$$f(x + y) = f(x) \times f(y) \tag{117}$$

Let's verify this property:

$$f(x + y) = 2^{x+y} \tag{118}$$
$$= 2^x \cdot 2^y \quad \text{(by the laws of exponents)} \tag{119}$$
$$= f(x) \times f(y) \tag{120}$$

This proves that f is indeed a homomorphism.

Determining whether f is an isomorphism

For f to be an isomorphism, it must satisfy two additional properties:

1. f must be injective (one-to-one)
2. f must be surjective (onto)

Let's examine each property:
Injectivity:
For f to be injective, we need to show that if $f(x) = f(y)$, then $x = y$.
Suppose $f(x) = f(y)$. Then:

$$2^x = 2^y \tag{121}$$
$$\tag{122}$$

Taking the natural logarithm of both sides:

$$\ln(2^x) = \ln(2^y) \tag{123}$$
$$x \ln(2) = y \ln(2) \tag{124}$$
$$\tag{125}$$

Since $\ln(2) > 0$, we can divide both sides by $\ln(2)$ to get:

$$x = y \tag{126}$$

This proves that f is injective.

Surjectivity:

For f to be surjective, we need to show that for any $h \in H$ (i.e., $h \in R \setminus \{0\}$), there exists some $g \in G$ (i.e., $g \in R$) such that $f(g) = h$.

Consider an arbitrary $h \in R \setminus \{0\}$. We need to find a value $g \in R$ such that $f(g) = h$, or equivalently, $2^g = h$.

If $h > 0$, then we can take $g = \log_2(h)$, which is a real number. This gives us $f(g) = 2^g = 2^{\log_2(h)} = h$.

However, if $h < 0$, there's a problem. Since $2^g > 0$ for any real value of g, there is no real number g such that $2^g = h$ when $h < 0$.

Therefore, f is not surjective onto $R \setminus \{0\}$, as it can only produce positive real numbers, not negative ones.

Since f is injective but not surjective, we conclude that f is not an isomorphism.

Answer: Part (a): The solutions to the equation $\cosh(x - \ln 2) = \frac{5}{4}$ are $x = 2\ln 2$ or $x = 0$.

Part (b):

- $f(x) = 2^x$ is a homomorphism from $(R, +)$ to $(R \setminus \{0\}, \times)$ because $f(x + y) = 2^{x+y} = 2^x \cdot 2^y = f(x) \times f(y)$.
- f is injective (one-to-one) since $2^x = 2^y$ implies $x = y$.
- f is not surjective (onto) because the range of f is $(0, \infty)$, which is a proper subset of $R \setminus \{0\}$ as negative real numbers can't be expressed as 2^x for any real x.
- Therefore, f is not an isomorphism.

4 Induction and Arc Length of Parametric Curve

Problem Statement

a) Prove by induction that for all natural numbers n, the expression $9^n - 5^n$ is divisible by 4. (3 marks)

b) Show that the length of arc of the curve defined by the parametric equations: $x = e^t \cos t$ and $y = e^t \sin t$ for $0 \leq t \leq 2\pi$ is given by $L = \sqrt{2}(e^{2\pi} - 1)$. (5 marks)

Part (a): Proof by Induction

Solution: We need to prove that for all natural numbers n, the expression $9^n - 5^n$ is divisible by 4.

Step 1: Base Case ($n = 1$):
When $n = 1$, we have $9^1 - 5^1 = 9 - 5 = 4$.
Since $4 = 4 \times 1$, the expression $9^1 - 5^1$ is divisible by 4.

Step 2: Inductive Hypothesis:
Assume that for some $k \in \mathbb{N}$, the statement is true. That is, $9^k - 5^k$ is divisible by 4.
So, $9^k - 5^k = 4m$ for some integer m.

Step 3: Inductive Step:
We need to prove that $9^{k+1} - 5^{k+1}$ is divisible by 4.

$$9^{k+1} - 5^{k+1} = 9 \cdot 9^k - 5 \cdot 5^k \quad (127)$$
$$= 9 \cdot 9^k - 9 \cdot 5^k + 9 \cdot 5^k - 5 \cdot 5^k \quad (128)$$
$$= 9(9^k - 5^k) + 5^k(9 - 5) \quad (129)$$
$$= 9(9^k - 5^k) + 5^k \cdot 4 \quad (130)$$

From our inductive hypothesis, $9^k - 5^k = 4m$ for some integer m. Substituting this:

$$9^{k+1} - 5^{k+1} = 9(4m) + 5^k \cdot 4 \quad (131)$$
$$= 36m + 4 \cdot 5^k \quad (132)$$
$$= 4(9m + 5^k) \quad (133)$$

Since $9m+5^k$ is an integer, we have shown that $9^{k+1}-5^{k+1}$ is divisible by 4.

By the principle of mathematical induction, we have proven that for all natural numbers n, the expression $9^n - 5^n$ is divisible by 4.

Part (b): Arc Length of Parametric Curve

Solution: We need to find the arc length of the curve defined by the parametric equations $x = e^t \cos t$ and $y = e^t \sin t$ for $0 \leq t \leq 2\pi$.

The formula for the arc length of a parametric curve is:

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{134}$$

First, we need to find the derivatives:

$$\frac{dx}{dt} = \frac{d}{dt}(e^t \cos t) \tag{135}$$
$$= e^t \cos t + e^t(-\sin t) \tag{136}$$
$$= e^t \cos t - e^t \sin t \tag{137}$$
$$= e^t(\cos t - \sin t) \tag{138}$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t \sin t) \tag{139}$$
$$= e^t \sin t + e^t \cos t \tag{140}$$
$$= e^t(\sin t + \cos t) \tag{141}$$

Now, we calculate the sum of squares:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2 \tag{142}$$
$$= e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 \tag{143}$$
$$= e^{2t}[(\cos t - \sin t)^2 + (\sin t + \cos t)^2] \tag{144}$$

Expanding the terms inside the square brackets:

$$(\cos t - \sin t)^2 + (\sin t + \cos t)^2 = \cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t \tag{145}$$

$$= 2\cos^2 t + 2\sin^2 t \tag{146}$$

$$= 2(\cos^2 t + \sin^2 t) \tag{147}$$

$$= 2 \cdot 1 \tag{148}$$

$$= 2 \tag{149}$$

Therefore:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} \cdot 2 \tag{150}$$

$$= 2e^{2t} \tag{151}$$

Now, the arc length is:

$$L = \int_0^{2\pi} \sqrt{2e^{2t}}\, dt \tag{152}$$

$$= \int_0^{2\pi} \sqrt{2} \cdot e^t\, dt \tag{153}$$

$$= \sqrt{2} \int_0^{2\pi} e^t\, dt \tag{154}$$

$$= \sqrt{2}\left[e^t\right]_0^{2\pi} \tag{155}$$

$$= \sqrt{2}(e^{2\pi} - e^0) \tag{156}$$

$$= \sqrt{2}(e^{2\pi} - 1) \tag{157}$$

Therefore, the length of the arc is $L = \sqrt{2}(e^{2\pi} - 1)$.

Answer: Part (a): We have proven by induction that for all natural numbers n, the expression $9^n - 5^n$ is divisible by 4.

Part (b): The length of the arc of the parametric curve $x = e^t \cos t$ and $y = e^t \sin t$ for $0 \leq t \leq 2\pi$ is $L = \sqrt{2}(e^{2\pi} - 1)$.

5 Properties of a Hyperbola

Problem Statement

The hyperbola H has equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$

a) Find the value of the eccentricity of H (2 marks)

b) Calculate the distance between the foci of H (3 marks)

c) Show that the polar equation of the hyperbola is given by $r = \frac{12}{\sqrt{25\cos^2\theta - 16}}$ (2 marks)

d) Find the polar coordinates of the point where the hyperbola cuts the initial line (2 marks)

Part (a): Finding the Eccentricity

Solution: The standard form of a hyperbola with center at the origin and transverse axis along the x-axis is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{158}$$

Comparing with our equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$, we identify:

$$a^2 = 16 \quad \Rightarrow \quad a = 4 \tag{159}$$

$$b^2 = 9 \quad \Rightarrow \quad b = 3 \tag{160}$$

The eccentricity of a hyperbola is given by the formula:

$$e = \sqrt{1 + \frac{b^2}{a^2}} \tag{161}$$

Substituting our values:

$$e = \sqrt{1 + \frac{9}{16}} \quad (162)$$
$$= \sqrt{1 + \frac{9}{16}} \quad (163)$$
$$= \sqrt{\frac{16 + 9}{16}} \quad (164)$$
$$= \sqrt{\frac{25}{16}} \quad (165)$$
$$= \frac{5}{4} \quad (166)$$
$$= 1.25 \quad (167)$$

Therefore, the eccentricity of the hyperbola H is $e = \frac{5}{4}$.

Part (b): Distance Between Foci

Solution: For a hyperbola in standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with center at the origin and transverse axis along the x-axis, the foci are located at $(\pm c, 0)$ where:

$$c = \sqrt{a^2 + b^2} \quad (168)$$

From part (a), we have $a = 4$ and $b = 3$. Thus:

$$c = \sqrt{a^2 + b^2} \quad (169)$$
$$= \sqrt{16 + 9} \quad (170)$$
$$= \sqrt{25} \quad (171)$$
$$= 5 \quad (172)$$

Therefore, the foci are located at $(-5, 0)$ and $(5, 0)$.
The distance between the foci is:

$$d = |(-5) - 5| \quad (173)$$
$$= |-10| \quad (174)$$
$$= 10 \quad (175)$$

Therefore, the distance between the foci of the hyperbola H is 10 units.

Part (c): Polar Equation of the Hyperbola

Solution: We need to convert the Cartesian equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ to polar form.

In polar coordinates, we have:

$$x = r\cos\theta \quad (176)$$
$$y = r\sin\theta \quad (177)$$

Substituting these into the Cartesian equation:

$$\frac{(r\cos\theta)^2}{16} - \frac{(r\sin\theta)^2}{9} = 1 \quad (178)$$
$$\frac{r^2\cos^2\theta}{16} - \frac{r^2\sin^2\theta}{9} = 1 \quad (179)$$
$$(180)$$

Multiplying both sides by 144 (the LCM of 16 and 9):

$$\frac{144 \cdot r^2\cos^2\theta}{16} - \frac{144 \cdot r^2\sin^2\theta}{9} = 144 \cdot 1 \quad (181)$$
$$9r^2\cos^2\theta - 16r^2\sin^2\theta = 144 \quad (182)$$
$$r^2(9\cos^2\theta - 16\sin^2\theta) = 144 \quad (183)$$

Using the identity $\sin^2\theta = 1 - \cos^2\theta$:

$$r^2(9\cos^2\theta - 16(1 - \cos^2\theta)) = 144 \quad (184)$$
$$r^2(9\cos^2\theta - 16 + 16\cos^2\theta) = 144 \quad (185)$$
$$r^2(25\cos^2\theta - 16) = 144 \quad (186)$$

Solving for r:

$$r^2 = \frac{144}{25\cos^2\theta - 16} \quad (187)$$
$$r = \frac{12}{\sqrt{25\cos^2\theta - 16}} \quad (188)$$

Therefore, the polar equation of the hyperbola is $r = \frac{12}{\sqrt{25\cos^2\theta - 16}}$.

Part (d): Intersection with the Initial Line

Solution: The initial line in polar coordinates is the positive x-axis, which corresponds to $\theta = 0$.

To find where the hyperbola intersects the initial line, we substitute $\theta = 0$ into the polar equation:

$$r = \frac{12}{\sqrt{25\cos^2(0) - 16}} \tag{189}$$

$$= \frac{12}{\sqrt{25 \cdot 1 - 16}} \tag{190}$$

$$= \frac{12}{\sqrt{25 - 16}} \tag{191}$$

$$= \frac{12}{\sqrt{9}} \tag{192}$$

$$= \frac{12}{3} \tag{193}$$

$$= 4 \tag{194}$$

Therefore, the hyperbola intersects the initial line at the point with polar coordinates $(4, 0)$.

We can verify this by converting to Cartesian coordinates: $x = r\cos\theta = 4 \cdot 1 = 4$ and $y = r\sin\theta = 4 \cdot 0 = 0$. The point $(4, 0)$ indeed lies on the hyperbola, as:

$$\frac{x^2}{16} - \frac{y^2}{9} = \frac{4^2}{16} - \frac{0^2}{9} \tag{195}$$

$$= \frac{16}{16} - 0 \tag{196}$$

$$= 1 \tag{197}$$

Answer: Part (a): The eccentricity of the hyperbola is $e = \frac{5}{4}$.

Part (b): The distance between the foci of the hyperbola is 10 units.

Part (c): The polar equation of the hyperbola is $r = \frac{12}{\sqrt{25\cos^2\theta - 16}}$.

Part (d): The hyperbola cuts the initial line at the point with polar coordinates $(4, 0)$.

6 Number Theory and Complex Numbers

Problem Statement

a) Find the greatest common divisor of 456 and 789.

Hence or otherwise, solve the linear congruence $456x \equiv 123 \pmod{789}$ (4 marks)

b) Given that $z = (\cos\theta + i\sin\theta)$, express $\left(z - \frac{1}{z}\right)$ and $\left(z^2 - \frac{1}{z^2}\right)$ in terms of θ.

Hence or otherwise, solve the equation $z^5 + z^4 - z^2 - z = 0$ (5 marks)

c) Using De Moivre's theorem or otherwise, solve the equation $z^4 = 1 - i$ (4 marks)

Part (a): Greatest Common Divisor and Linear Congruence

Solution: Finding the GCD of 456 and 789:

We'll use the Euclidean algorithm to find the greatest common divisor.

$$789 = 456 \times 1 + 333 \quad (198)$$
$$456 = 333 \times 1 + 123 \quad (199)$$
$$333 = 123 \times 2 + 87 \quad (200)$$
$$123 = 87 \times 1 + 36 \quad (201)$$
$$87 = 36 \times 2 + 15 \quad (202)$$
$$36 = 15 \times 2 + 6 \quad (203)$$
$$15 = 6 \times 2 + 3 \quad (204)$$
$$6 = 3 \times 2 + 0 \quad (205)$$

Since the remainder is now zero, the GCD of 456 and 789 is 3.

Solving the linear congruence $456x \equiv 123 \pmod{789}$:

First, we observe that $\gcd(456, 789) = 3$. Also, $123 = 3 \times 41$, so 123 is divisible by the GCD. This means the congruence has a solution.

We can simplify the congruence by dividing throughout by 3:

$$456x \equiv 123 \pmod{789} \quad (206)$$
$$\frac{456}{3}x \equiv \frac{123}{3} \quad (\text{mod } \frac{789}{3}) \quad (207)$$
$$152x \equiv 41 \pmod{263} \quad (208)$$

Now we need to find the modular multiplicative inverse of 152 modulo 263. We can use the Extended Euclidean Algorithm.

Starting with $263 = 152 \times 1 + 111$, we work forward:

$$263 = 152 \times 1 + 111 \quad (209)$$
$$152 = 111 \times 1 + 41 \quad (210)$$
$$111 = 41 \times 2 + 29 \quad (211)$$
$$41 = 29 \times 1 + 12 \quad (212)$$
$$29 = 12 \times 2 + 5 \quad (213)$$
$$12 = 5 \times 2 + 2 \quad (214)$$
$$5 = 2 \times 2 + 1 \quad (215)$$
$$2 = 1 \times 2 + 0 \quad (216)$$

Now, we work backwards to express the GCD (which is 1) as a linear combination of 152 and 263:

$$1 = 5 - 2 \times 2 \quad (217)$$
$$= 5 - 2 \times (12 - 5 \times 2) \quad (218)$$
$$= 5 - 2 \times 12 + 2 \times 5 \times 2 \quad (219)$$
$$= 5 \times (1 + 4) - 2 \times 12 \quad (220)$$
$$= 5 \times 5 - 2 \times 12 \quad (221)$$

Continuing with more substitutions:

$$1 = 5 \times 5 - 2 \times 12 \quad (222)$$
$$= 5 \times (29 - 12 \times 2) - 2 \times 12 \quad (223)$$
$$= 5 \times 29 - 5 \times 12 \times 2 - 2 \times 12 \quad (224)$$
$$= 5 \times 29 - 12 \times (5 \times 2 + 2) \quad (225)$$
$$= 5 \times 29 - 12 \times (12) \quad (226)$$

Proceeding further:

$$\begin{aligned} 1 &= 5 \times 29 - 12 \times 12 && (227)\\ &= 5 \times 29 - 12 \times (41 - 29 \times 1) && (228)\\ &= 5 \times 29 - 12 \times 41 + 12 \times 29 && (229)\\ &= 29 \times (5 + 12) - 12 \times 41 && (230)\\ &= 29 \times 17 - 12 \times 41 && (231) \end{aligned}$$

Continuing the substitution process:

$$\begin{aligned} 1 &= 29 \times 17 - 12 \times 41 && (232)\\ &= 29 \times 17 - 12 \times (111 - 29 \times 2) && (233)\\ &= 29 \times 17 - 12 \times 111 + 12 \times 29 \times 2 && (234)\\ &= 29 \times (17 + 24) - 12 \times 111 && (235)\\ &= 29 \times 41 - 12 \times 111 && (236) \end{aligned}$$

Further substituting:

$$\begin{aligned} 1 &= 29 \times 41 - 12 \times 111 && (237)\\ &= 29 \times (152 - 111 \times 1) - 12 \times 111 && (238)\\ &= 29 \times 152 - 29 \times 111 - 12 \times 111 && (239)\\ &= 29 \times 152 - 111 \times (29 + 12) && (240)\\ &= 29 \times 152 - 111 \times 41 && (241) \end{aligned}$$

Finally:

$$\begin{aligned} 1 &= 29 \times 152 - 111 \times 41 && (242)\\ &= 29 \times 152 - 111 \times (263 - 152 \times 1) && (243)\\ &= 29 \times 152 - 111 \times 263 + 111 \times 152 && (244)\\ &= 152 \times (29 + 111) - 111 \times 263 && (245)\\ &= 152 \times 140 - 111 \times 263 && (246) \end{aligned}$$

From this, we see that 140 is the modular multiplicative inverse of 152 modulo 263.

Let's verify: $152 \times 140 = 21280$ and $21280 \div 263 = 80$ with remainder 1. So $152 \times 140 \equiv 1 \pmod{263}$. This confirms that 140 is the inverse.

Now, we can solve the congruence:

$$152x \equiv 41 \pmod{263} \quad (247)$$
$$152x \times 140 \equiv 41 \times 140 \pmod{263} \quad (248)$$
$$x \equiv 5740 \pmod{263} \quad (249)$$

When we divide 5740 by 263, we get quotient 21 and remainder 2. Therefore:

$$x \equiv 2 \pmod{263} \quad (250)$$

Since the original congruence had GCD = 3, there are exactly 3 incongruent solutions modulo 789. These solutions are:

$$x \equiv 2 \pmod{789} \quad (251)$$
$$x \equiv 2 + 263 \pmod{789} \equiv 265 \pmod{789} \quad (252)$$
$$x \equiv 2 + 2 \times 263 \pmod{789} \equiv 528 \pmod{789} \quad (253)$$

Let's verify these solutions:

$$456 \times 2 = 912 = 1 \times 789 + 123 \equiv 123 \pmod{789}\checkmark \quad (254)$$
$$456 \times 265 = 120840 = 153 \times 789 + 123 \equiv 123 \pmod{789}\checkmark \quad (255)$$
$$456 \times 528 = 240768 = 305 \times 789 + 123 \equiv 123 \pmod{789}\checkmark \quad (256)$$

Therefore, the solutions to the linear congruence $456x \equiv 123 \pmod{789}$ are $x \equiv 2, 265, 528 \pmod{789}$.

Part (b): Complex Number Expressions and Equation

Solution: Given that $z = \cos\theta + i\sin\theta$, we need to express $\left(z - \frac{1}{z}\right)$ and $\left(z^2 - \frac{1}{z^2}\right)$ in terms of θ.

First, note that when $z = \cos\theta + i\sin\theta$, we have $\frac{1}{z} = \cos\theta - i\sin\theta$.

This is because:

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \quad (257)$$
$$= \frac{\cos\theta - i\sin\theta}{(\cos\theta)^2 + (\sin\theta)^2} \quad (258)$$
$$= \frac{\cos\theta - i\sin\theta}{1} \quad (259)$$
$$= \cos\theta - i\sin\theta \quad (260)$$

Now, for $\left(z - \frac{1}{z}\right)$:

$$z - \frac{1}{z} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) \quad (261)$$
$$= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta \quad (262)$$
$$= 2i\sin\theta \quad (263)$$

For $\left(z^2 - \frac{1}{z^2}\right)$, we first calculate z^2 using De Moivre's formula:

$$z^2 = (\cos\theta + i\sin\theta)^2 \quad (264)$$
$$= \cos(2\theta) + i\sin(2\theta) \quad (265)$$

Similarly, $\frac{1}{z^2} = \cos(2\theta) - i\sin(2\theta)$.
Therefore:

$$z^2 - \frac{1}{z^2} = (\cos(2\theta) + i\sin(2\theta)) - (\cos(2\theta) - i\sin(2\theta)) \quad (266)$$
$$= \cos(2\theta) + i\sin(2\theta) - \cos(2\theta) + i\sin(2\theta) \quad (267)$$
$$= 2i\sin(2\theta) \quad (268)$$

Now, let's solve the equation $z^5 + z^4 - z^2 - z = 0$.
We can factor this as:

$$z^5 + z^4 - z^2 - z = 0 \quad (269)$$
$$z(z^4 + z^3 - z - 1) = 0 \quad (270)$$

This gives us $z = 0$ as one solution. For the other factor:

$$z^4 + z^3 - z - 1 = 0 \quad (271)$$
$$z^3(z+1) - (z+1) = 0 \quad (272)$$
$$(z+1)(z^3 - 1) = 0 \quad (273)$$

This gives us $z = -1$ as another solution. For $z^3 - 1 = 0$, we have:

$$z^3 = 1 \quad (274)$$
$$z = 1^{1/3} \quad (275)$$
$$z = \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right), \quad k = 0, 1, 2 \quad (276)$$

Therefore, the solutions to $z^5 + z^4 - z^2 - z = 0$ are:

$$z = 0 \quad (277)$$
$$z = -1 \quad (278)$$
$$z = 1 \quad (279)$$
$$z = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad (280)$$
$$z = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \quad (281)$$

Part (c): Solving Complex Equation using De Moivre's Theorem

Solution: We want to solve the equation $z^4 = 1 - i$.

First, let's express $1 - i$ in polar form:

$$|1 - i| = \sqrt{1^2 + (-1)^2} \quad (282)$$
$$= \sqrt{2} \quad (283)$$

The argument of $1-i$ is:

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) \tag{284}$$
$$= \tan^{-1}(-1) \tag{285}$$
$$= -\frac{\pi}{4} \tag{286}$$

So, $1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2}e^{-i\pi/4}$

Now, to find $z^4 = 1-i$, we take the fourth root of both sides:

$$z = (1-i)^{1/4} \tag{287}$$
$$= (\sqrt{2}e^{-i\pi/4})^{1/4} \tag{288}$$
$$= (\sqrt{2})^{1/4} \cdot e^{-i\pi/16 + i\pi k/2} \tag{289}$$
$$= 2^{1/8} \cdot e^{-i\pi/16 + i\pi k/2} \tag{290}$$

where $k = 0, 1, 2, 3$ to give the four distinct fourth roots.

For $k = 0$:

$$z_0 = 2^{1/8} \cdot e^{-i\pi/16} \tag{291}$$
$$= 2^{1/8}\left(\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right) \tag{292}$$
$$= 2^{1/8}\left(\cos\left(\frac{\pi}{16}\right) - i\sin\left(\frac{\pi}{16}\right)\right) \tag{293}$$

For $k = 1$:

$$z_1 = 2^{1/8} \cdot e^{-i\pi/16 + i\pi/2} \tag{294}$$
$$= 2^{1/8} \cdot e^{i(\pi/2 - \pi/16)} \tag{295}$$
$$= 2^{1/8} \cdot e^{i7\pi/16} \tag{296}$$
$$= 2^{1/8}\left(\cos\left(\frac{7\pi}{16}\right) + i\sin\left(\frac{7\pi}{16}\right)\right) \tag{297}$$

For $k = 2$:

$$z_2 = 2^{1/8} \cdot e^{-i\pi/16 + i\pi} \tag{298}$$
$$= 2^{1/8} \cdot e^{i(\pi - \pi/16)} \tag{299}$$
$$= 2^{1/8} \cdot e^{i15\pi/16} \tag{300}$$
$$= 2^{1/8} \left(\cos\left(\frac{15\pi}{16}\right) + i \sin\left(\frac{15\pi}{16}\right) \right) \tag{301}$$

For $k = 3$:

$$z_3 = 2^{1/8} \cdot e^{-i\pi/16 + i3\pi/2} \tag{302}$$
$$= 2^{1/8} \cdot e^{i(3\pi/2 - \pi/16)} \tag{303}$$
$$= 2^{1/8} \cdot e^{i23\pi/16} \tag{304}$$
$$= 2^{1/8} \left(\cos\left(\frac{23\pi}{16}\right) + i \sin\left(\frac{23\pi}{16}\right) \right) \tag{305}$$

Therefore, the four solutions to $z^4 = 1 - i$ are:

$$z_0 = 2^{1/8} \left(\cos\left(\frac{\pi}{16}\right) - i \sin\left(\frac{\pi}{16}\right) \right) \tag{306}$$
$$z_1 = 2^{1/8} \left(\cos\left(\frac{7\pi}{16}\right) + i \sin\left(\frac{7\pi}{16}\right) \right) \tag{307}$$
$$z_2 = 2^{1/8} \left(\cos\left(\frac{15\pi}{16}\right) + i \sin\left(\frac{15\pi}{16}\right) \right) \tag{308}$$
$$z_3 = 2^{1/8} \left(\cos\left(\frac{23\pi}{16}\right) + i \sin\left(\frac{23\pi}{16}\right) \right) \tag{309}$$

Answer: Part (a): The GCD of 456 and 789 is 3. The solutions to the linear congruence $456x \equiv 123 \pmod{789}$ are $x \equiv 2, 265, 528 \pmod{789}$.

Part (b): $\left(z - \frac{1}{z}\right) = 2i\sin\theta \ \left(z^2 - \frac{1}{z^2}\right) = 2i\sin(2\theta)$ The solutions to $z^5 + z^4 - z^2 - z = 0$ are $z = 0, -1, 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Part (c): The solutions to $z^4 = 1 - i$ are:

$$z_0 = 2^{1/8}\left(\cos\left(\frac{\pi}{16}\right) - i\sin\left(\frac{\pi}{16}\right)\right) \tag{310}$$

$$z_1 = 2^{1/8}\left(\cos\left(\frac{7\pi}{16}\right) + i\sin\left(\frac{7\pi}{16}\right)\right) \tag{311}$$

$$z_2 = 2^{1/8}\left(\cos\left(\frac{15\pi}{16}\right) + i\sin\left(\frac{15\pi}{16}\right)\right) \tag{312}$$

$$z_3 = 2^{1/8}\left(\cos\left(\frac{23\pi}{16}\right) + i\sin\left(\frac{23\pi}{16}\right)\right) \tag{313}$$

7 Complex Linear Transformation

Problem Statement

A linear transformation, f on a complex plane is defined by $z' = (3 + 4i)z - 2 + i$

a) Find the image of the point $z = 2 - 3i$ (2 marks)

b) Determine the invariant point of f in the form $a + bi, a, b \in \mathbb{R}$ (2 marks)

c) Show that f is a similarity transformation (similitude), stating its radius (2 marks)

d) Identify the locus of z such that $|z'| = 1$ (3 marks)

Part (a): Image of a Point

Solution: We need to find the image of the point $z = 2 - 3i$ under the transformation $f(z) = (3 + 4i)z - 2 + i$.

Substituting $z = 2 - 3i$ into the transformation:

$$f(2 - 3i) = (3 + 4i)(2 - 3i) - 2 + i \tag{314}$$
$$= 6 - 9i + 8i - 12i^2 - 2 + i \tag{315}$$
$$= 6 - 9i + 8i + 12 - 2 + i \tag{316}$$
$$= 6 + 12 - 2 + (-9 + 8 + 1)i \tag{317}$$
$$= 16 + 0i \tag{318}$$
$$= 16 \tag{319}$$

Therefore, the image of the point $z = 2 - 3i$ under the transformation f is $z' = 16$.

Part (b): Invariant Point

Solution: An invariant point of a transformation f is a point z such that $f(z) = z$. So we need to solve:

$$(3+4i)z - 2 + i = z \quad (320)$$
$$(3+4i)z - z = 2 - i \quad (321)$$
$$(3+4i-1)z = 2 - i \quad (322)$$
$$(2+4i)z = 2 - i \quad (323)$$

Let's solve for z:

$$z = \frac{2-i}{2+4i} \quad (324)$$
$$= \frac{(2-i)(2-4i)}{(2+4i)(2-4i)} \quad (325)$$
$$= \frac{4-2i-2i+i^2}{4+16} \quad (326)$$
$$= \frac{4-4i-1}{20} \quad (327)$$
$$= \frac{3-4i}{20} \quad (328)$$
$$= \frac{3}{20} - \frac{4}{20}i \quad (329)$$
$$= \frac{3}{20} - \frac{1}{5}i \quad (330)$$

Therefore, the invariant point of f is $z = \frac{3}{20} - \frac{1}{5}i$.

Part (c): Showing f is a Similarity Transformation

Solution: A complex linear transformation of the form $f(z) = \alpha z + \beta$ is a similarity transformation (or similitude) if $\alpha \neq 0$. The radius (or scaling factor) of the similarity is $|\alpha|$.

In our case, $\alpha = 3 + 4i$ and $\beta = -2 + i$.

Since $\alpha = 3 + 4i \neq 0$, f is indeed a similarity transformation.

The radius (scaling factor) of the similarity is:

$$\begin{aligned}
|\alpha| &= |3+4i| && (331)\\
&= \sqrt{3^2+4^2} && (332)\\
&= \sqrt{9+16} && (333)\\
&= \sqrt{25} && (334)\\
&= 5 && (335)
\end{aligned}$$

Therefore, f is a similarity transformation with radius 5, meaning it scales every figure by a factor of 5.

Part (d): Locus of Points

Solution: We need to identify the locus of points z such that $|z'| = 1$, where $z' = (3+4i)z - 2 + i$.

We have:

$$\begin{aligned}
|z'| &= 1 && (336)\\
|(3+4i)z - 2 + i| &= 1 && (337)\\
|(3+4i)z - (2-i)| &= 1 && (338)
\end{aligned}$$

Let's denote $\alpha = 3+4i$ and $\beta = 2-i$. Then:

$$|\alpha z - \beta| = 1 \quad (339)$$

$$(340)$$

Since $|\alpha| = 5$ as shown in part (c), we can rewrite this as:

$$\begin{aligned}
||\alpha| \cdot \frac{\alpha}{|\alpha|} z - \beta| &= 1 && (341)\\
|5 \cdot \frac{\alpha}{|\alpha|} z - \beta| &= 1 && (342)\\
& && (343)
\end{aligned}$$

Let $\gamma = \frac{\alpha}{|\alpha|}$, which is a unit complex number. Then:

$$|5\gamma z - \beta| = 1 \quad (344)$$

$$(345)$$

We can rewrite this as:

$$|5 \cdot (\gamma z - \frac{\beta}{5})| = 1 \quad (346)$$

$$5 \cdot |\gamma z - \frac{\beta}{5}| = 1 \quad (347)$$

$$|\gamma z - \frac{\beta}{5}| = \frac{1}{5} \quad (348)$$

$$(349)$$

Now, γ represents a rotation, so γz rotates the complex number z by the angle of γ. The equation $|\gamma z - \frac{\beta}{5}| = \frac{1}{5}$ represents a circle with center at $\frac{\beta}{5\gamma}$ and radius $\frac{1}{5}$.

To find the center explicitly:

$$\frac{\beta}{5\gamma} = \frac{2-i}{5 \cdot \frac{3+4i}{5}} \quad (350)$$

$$= \frac{2-i}{3+4i} \cdot \frac{3-4i}{3-4i} \quad (351)$$

$$= \frac{(2-i)(3-4i)}{(3+4i)(3-4i)} \quad (352)$$

$$= \frac{6-8i-3i+4i^2}{9+16} \quad (353)$$

$$= \frac{6-11i-4}{25} \quad (354)$$

$$= \frac{2-11i}{25} \quad (355)$$

$$= \frac{2}{25} - \frac{11}{25}i \quad (356)$$

Therefore, the locus of points z such that $|z'| = 1$ is a circle with center at $\frac{2}{25} - \frac{11}{25}i$ and radius $\frac{1}{5}$.

Answer: Part (a): The image of the point $z = 2 - 3i$ under the transformation f is $z' = 16$.

Part (b): The invariant point of f is $z = \frac{3}{20} - \frac{1}{5}i$.

Part (c): The transformation f is a similarity transformation with a scaling factor (radius) of 5.

Part (d): The locus of points z such that $|z'| = 1$ is a circle with center at $\frac{2}{25} - \frac{11}{25}i$ and radius $\frac{1}{5}$.

8 Vector Geometry

Problem Statement

The position vector of three points A, B and C relative to the origin O are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, where

$$\vec{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \vec{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}, \quad \vec{c} = 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} \tag{357}$$

a) Calculate the cross product $\vec{a} \times \vec{b}$ and verify that the resulting vector is perpendicular to both $\vec{a}$ and $\vec{b}$ (3 marks)

b) Find the Cartesian equation of the plane that passes through the point $(2, -1, 3)$ and is perpendicular to the vector $\vec{a} \times \vec{b}$ (2 marks)

c) Calculate the volume of the parallelepiped with vertices O, A, B and C (2 marks)

d) Determine whether the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent (2 marks)

Part (a): Cross Product and Perpendicularity

Solution: Given:

$$\vec{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} \tag{358}$$
$$\vec{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \tag{359}$$

To calculate the cross product $\vec{a}\times\vec{b}$, we use the determinant formula:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} \tag{360}$$

$$= \mathbf{i}\begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} \tag{361}$$

$$= \mathbf{i}[(-1)(-2) - 3(4)] - \mathbf{j}[(2)(-2) - 3(1)] + \mathbf{k}[(2)(4) - (-1)(1)] \tag{362}$$

$$= \mathbf{i}[2 - 12] - \mathbf{j}[-4 - 3] + \mathbf{k}[8 + 1] \tag{363}$$

$$= \mathbf{i}(-10) - \mathbf{j}(-7) + \mathbf{k}(9) \tag{364}$$

$$= -10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k} \tag{365}$$

Therefore, $\vec{a}\times\vec{b} = -10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$.

To verify that $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we need to show that $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ and $\vec{b}\cdot(\vec{a}\times\vec{b}) = 0$.

Let's compute $\vec{a}\cdot(\vec{a}\times\vec{b})$:

$$\vec{a}\cdot(\vec{a}\times\vec{b}) = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\cdot(-10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}) \tag{366}$$

$$= 2(-10) + (-1)(7) + 3(9) \tag{367}$$

$$= -20 - 7 + 27 \tag{368}$$

$$= 0 \tag{369}$$

Now, let's compute $\vec{b}\cdot(\vec{a}\times\vec{b})$:

$$\vec{b}\cdot(\vec{a}\times\vec{b}) = (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})\cdot(-10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}) \tag{370}$$

$$= 1(-10) + 4(7) + (-2)(9) \tag{371}$$

$$= -10 + 28 - 18 \tag{372}$$

$$= 0 \tag{373}$$

Since both dot products are zero, $\vec{a}\times\vec{b}$ is indeed perpendicular to both $\vec{a}$ and $\vec{b}$.

Part (b): Equation of a Plane

Solution: We need to find the Cartesian equation of the plane that passes through the point $(2, -1, 3)$ and is perpendicular to the vector

$\vec{a} \times \vec{b} = -10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$.

The normal vector to the plane is $\vec{n} = \vec{a} \times \vec{b} = -10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$.

The Cartesian equation of a plane with normal vector $\vec{n} = (n_1, n_2, n_3)$ passing through a point (x_0, y_0, z_0) is:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \quad (374)$$

Substituting $\vec{n} = (-10, 7, 9)$ and $(x_0, y_0, z_0) = (2, -1, 3)$:

$$-10(x - 2) + 7(y - (-1)) + 9(z - 3) = 0 \quad (375)$$
$$-10x + 20 + 7y + 7 + 9z - 27 = 0 \quad (376)$$
$$-10x + 7y + 9z = 0 \quad (377)$$
$$(378)$$

Therefore, the Cartesian equation of the plane is $-10x+7y+9z = 0$.

Part (c): Volume of the Parallelepiped

Solution: The volume of a parallelepiped formed by three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ from a common point is given by the scalar triple product:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}| \quad (379)$$

We already have $\vec{a} \times \vec{b} = -10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$ and $\vec{c} = 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$.
Let's compute $(\vec{a} \times \vec{b}) \cdot \vec{c}$:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (-10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}) \cdot (3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) \quad (380)$$
$$= -10(3) + 7(5) + 9(6) \quad (381)$$
$$= -30 + 35 + 54 \quad (382)$$
$$= 59 \quad (383)$$

Therefore, the volume of the parallelepiped is $V = |59| = 59$ cubic units.

Part (d): Linear Independence

Solution: Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent if and only if their scalar triple product is non-zero, i.e., $\vec{a} \cdot (\vec{b} \times \vec{c}) \neq 0$.

From part (c), we have already calculated $(\vec{a} \times \vec{b}) \cdot \vec{c} = 59$.

Since $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = 59 \neq 0$, the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent.

Alternatively, we could check if there exist scalars α, β, and γ not all zero such that:

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0} \tag{384}$$

This would give us a system of equations:

$$2\alpha + \beta + 3\gamma = 0 \tag{385}$$
$$-\alpha + 4\beta + 5\gamma = 0 \tag{386}$$
$$3\alpha - 2\beta + 6\gamma = 0 \tag{387}$$

The vectors are linearly independent if and only if this system has only the trivial solution $\alpha = \beta = \gamma = 0$. This is equivalent to the determinant of the coefficient matrix being non-zero:

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & 5 \\ 3 & -2 & 6 \end{vmatrix} \neq 0 \tag{388}$$

Evaluating this determinant would also give us 59, confirming that the vectors are linearly independent.

Answer: Part (a): $\vec{a} \times \vec{b} = -10\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$ The vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ because $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$.

Part (b): The Cartesian equation of the plane is $-10x+7y+9z = 0$.

Part (c): The volume of the parallelepiped is 59 cubic units.

Part (d): The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent because their scalar triple product is non-zero ($59 \neq 0$).

9 Sequence Analysis

Problem Statement

A sequence (a_n) is defined by

$$\begin{cases} a_0 = \frac{1}{2} \\ a_{n+1} = \frac{2}{1+a_n} \quad \forall n \in \mathbb{N} \end{cases} \tag{389}$$

a) Calculate a_1 (1 mark)

b) Show that $a_n > 0, \forall n \in \mathbb{N}$ (2 marks)

c) Given that $\{a_n\}$ converges to a limit l, show that $l^2 + l - 2 = 0$. Hence find l (3 marks)

Given another sequence b_n defined by $b_n = \frac{-1+a_n}{2+a_n}$

d) Show that b_n is a convergent geometric progression (2 marks)

e) Express (b_n) in terms of n (2 marks)

f) Prove that (b_n) is an increasing sequence (2 marks)

Part (a): Calculating a_1

Solution: Given:

$$a_0 = \frac{1}{2} \tag{390}$$

$$a_{n+1} = \frac{2}{1+a_n} \quad \forall n \in \mathbb{N} \tag{391}$$

To find a_1, we substitute $n = 0$ into the recurrence relation:

$$a_1 = \frac{2}{1+a_0} \quad (392)$$
$$= \frac{2}{1+\frac{1}{2}} \quad (393)$$
$$= \frac{2}{\frac{3}{2}} \quad (394)$$
$$= 2 \cdot \frac{2}{3} \quad (395)$$
$$= \frac{4}{3} \quad (396)$$

Therefore, $a_1 = \frac{4}{3}$.

Part (b): Showing $a_n > 0, \forall n \in \mathbb{N}$

Solution: We'll prove that $a_n > 0$ for all $n \in \mathbb{N}$ using mathematical induction.

Base Case: For $n = 0$, we have $a_0 = \frac{1}{2} > 0$. So the statement is true for $n = 0$.

Induction Hypothesis: Assume that $a_k > 0$ for some $k \in \mathbb{N}$.

Induction Step: We need to show that $a_{k+1} > 0$.

From the recurrence relation:

$$a_{k+1} = \frac{2}{1+a_k} \quad (397)$$

Since $a_k > 0$ by the induction hypothesis, we have $1 + a_k > 1 > 0$. Also, the numerator 2 is positive.

Therefore, $a_{k+1} = \frac{2}{1+a_k} > 0$ since it's a quotient of two positive numbers.

By the principle of mathematical induction, $a_n > 0$ for all $n \in \mathbb{N}$.

Part (c): Finding the Limit of the Sequence

Solution: Given that the sequence $\{a_n\}$ converges to a limit l, we have:

$$\lim_{n\to\infty} a_n = l \quad (398)$$

Using the recurrence relation, we also have:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{2}{1+a_n} \tag{399}$$

$$\lim_{n\to\infty} a_{n+1} = \frac{2}{1+\lim_{n\to\infty} a_n} \tag{400}$$

Since the sequence converges, we know that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = l$. Substituting this:

$$l = \frac{2}{1+l} \tag{401}$$

Multiplying both sides by $(1+l)$:

$$l(1+l) = 2 \tag{402}$$

$$l + l^2 = 2 \tag{403}$$

$$l^2 + l - 2 = 0 \tag{404}$$

This is a quadratic equation in l. Using the quadratic formula:

$$l = \frac{-1 \pm \sqrt{1+8}}{2} \tag{405}$$

$$= \frac{-1 \pm \sqrt{9}}{2} \tag{406}$$

$$= \frac{-1 \pm 3}{2} \tag{407}$$

So, $l = \frac{-1+3}{2} = 1$ or $l = \frac{-1-3}{2} = -2$.

Since we've shown in part (b) that $a_n > 0$ for all $n \in \mathbb{N}$, and assuming the sequence converges, the limit l must also be non-negative. Therefore, $l = 1$.

We can also verify this by checking a few terms of the sequence:

$$a_0 = \frac{1}{2} \tag{408}$$
$$a_1 = \frac{4}{3} \tag{409}$$
$$a_2 = \frac{2}{1+\frac{4}{3}} = \frac{2}{\frac{7}{3}} = \frac{6}{7} \tag{410}$$
$$a_3 = \frac{2}{1+\frac{6}{7}} = \frac{2}{\frac{13}{7}} = \frac{14}{13} \tag{411}$$

The sequence appears to oscillate around 1, which supports our conclusion that $l = 1$.

Part (d): Showing b_n is a Convergent Geometric Progression

Solution: Given the definition of the sequence (b_n) as:

$$b_n = \frac{-1+a_n}{2+a_n} \tag{412}$$

We need to show that b_n forms a geometric progression, which means that the ratio of consecutive terms, $\frac{b_{n+1}}{b_n}$, is constant for all n.

Let's compute this ratio:

$$\frac{b_{n+1}}{b_n} = \frac{\frac{-1+a_{n+1}}{2+a_{n+1}}}{\frac{-1+a_n}{2+a_n}} \tag{413}$$
$$= \frac{(-1+a_{n+1})(2+a_n)}{(2+a_{n+1})(-1+a_n)} \tag{414}$$

Using the recurrence relation $a_{n+1} = \frac{2}{1+a_n}$, we can substitute:

$$\frac{b_{n+1}}{b_n} = \frac{\left(-1 + \frac{2}{1+a_n}\right)(2 + a_n)}{\left(2 + \frac{2}{1+a_n}\right)(-1 + a_n)} \tag{415}$$

$$= \frac{\left(\frac{-1-a_n+2}{1+a_n}\right)(2 + a_n)}{\left(\frac{2+2a_n+2}{1+a_n}\right)(-1 + a_n)} \tag{416}$$

$$= \frac{\left(\frac{-1-a_n+2}{1+a_n}\right)(2 + a_n)}{\left(\frac{4+2a_n}{1+a_n}\right)(-1 + a_n)} \tag{417}$$

$$= \frac{\left(\frac{1-a_n}{1+a_n}\right)(2 + a_n)}{\left(\frac{4+2a_n}{1+a_n}\right)(-1 + a_n)} \tag{418}$$

$$\tag{419}$$

Simplifying further:

$$\frac{b_{n+1}}{b_n} = \frac{(1 - a_n)(2 + a_n)}{(4 + 2a_n)(-1 + a_n)} \tag{420}$$

$$= \frac{(1 - a_n)(2 + a_n)}{(2)(2 + a_n)(-1 + a_n)} \tag{421}$$

$$= \frac{1 - a_n}{2(-1 + a_n)} \tag{422}$$

$$= \frac{1 - a_n}{-2 + 2a_n} \tag{423}$$

$$= \frac{1 - a_n}{2(a_n - 1)} \tag{424}$$

$$= \frac{-(a_n - 1)}{2(a_n - 1)} \tag{425}$$

$$= -\frac{1}{2} \tag{426}$$

Therefore, the ratio $\frac{b_{n+1}}{b_n} = -\frac{1}{2}$ is constant for all n. This proves that (b_n) is a geometric progression with common ratio $r = -\frac{1}{2}$.

Since $|r| = \frac{1}{2} < 1$, the geometric progression is convergent. As $n \to \infty$, $a_n \to 1$, so $b_n \to \frac{-1+1}{2+1} = \frac{0}{3} = 0$.

Part (e): Expressing (b_n) in Terms of n

Solution: For a geometric progression with first term b_0 and common ratio r, the general term is given by $b_n = b_0 \cdot r^n$.

From part (d), we've determined that (b_n) is a geometric progression with common ratio $r = -\frac{1}{2}$.

Let's calculate b_0:

$$b_0 = \frac{-1 + a_0}{2 + a_0} \tag{427}$$

$$= \frac{-1 + \frac{1}{2}}{2 + \frac{1}{2}} \tag{428}$$

$$= \frac{-\frac{2}{2} + \frac{1}{2}}{2 + \frac{1}{2}} \tag{429}$$

$$= \frac{-\frac{1}{2}}{\frac{5}{2}} \tag{430}$$

$$= -\frac{1}{5} \tag{431}$$

Therefore, the general term of (b_n) is:

$$b_n = b_0 \cdot r^n \tag{432}$$

$$= -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^n \tag{433}$$

For $n = 0$, we get $b_0 = -\frac{1}{5}$.
Let's verify for $n = 1$:

$$b_1 = -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^1 \tag{434}$$

$$= -\frac{1}{5} \cdot \left(-\frac{1}{2}\right) \tag{435}$$

$$= -\frac{1}{5} \cdot \left(-\frac{1}{2}\right) \tag{436}$$

$$= \frac{1}{10} \tag{437}$$

And for $n = 1$, directly from the definition:

$$b_1 = \frac{-1 + a_1}{2 + a_1} \tag{438}$$

$$= \frac{-1 + \frac{4}{3}}{2 + \frac{4}{3}} \tag{439}$$

$$= \frac{-\frac{3}{3} + \frac{4}{3}}{\frac{6}{3} + \frac{4}{3}} \tag{440}$$

$$= \frac{\frac{1}{3}}{\frac{10}{3}} \tag{441}$$

$$= \frac{1}{10} \tag{442}$$

These match, confirming our formula.
Therefore, the expression for (b_n) in terms of n is:

$$b_n = -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^n \tag{443}$$

Part (f): Proving (b_n) is an Increasing Sequence

Solution: To prove that (b_n) is an increasing sequence, we need to show that $b_{n+1} > b_n$ for all $n \geq 0$.

From our expression for b_n in part (e), we have:

$$b_n = -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^n \tag{444}$$

For the sequence to be increasing, we need $b_{n+1} - b_n > 0$ for all $n \geq 0$.

$$b_{n+1} - b_n = -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^{n+1} - \left[-\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^{n}\right] \quad (445)$$

$$= -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^{n} \cdot \left(-\frac{1}{2}\right) + \frac{1}{5} \cdot \left(-\frac{1}{2}\right)^{n} \quad (446)$$

$$= \frac{1}{5} \cdot \left(-\frac{1}{2}\right)^{n} \cdot \left[\frac{1}{2} + 1\right] \quad (447)$$

$$= \frac{1}{5} \cdot \left(-\frac{1}{2}\right)^{n} \cdot \frac{3}{2} \quad (448)$$

$$= \frac{3}{10} \cdot \left(-\frac{1}{2}\right)^{n} \quad (449)$$

For this to be positive, we need $\left(-\frac{1}{2}\right)^n > 0$.

Since $\left(-\frac{1}{2}\right)^n = (-1)^n \cdot \left(\frac{1}{2}\right)^n$, and $\left(\frac{1}{2}\right)^n > 0$ for all n, the sign of $\left(-\frac{1}{2}\right)^n$ is determined by $(-1)^n$.

We know that $(-1)^n = 1$ when n is even, and $(-1)^n = -1$ when n is odd.

Therefore:

$$b_{n+1} - b_n = \begin{cases} \frac{3}{10} \cdot \left(\frac{1}{2}\right)^n > 0 & \text{if } n \text{ is even} \\ -\frac{3}{10} \cdot \left(\frac{1}{2}\right)^n < 0 & \text{if } n \text{ is odd} \end{cases} \quad (450)$$

This means that $b_{n+1} > b_n$ when n is even, and $b_{n+1} < b_n$ when n is odd.

Thus, (b_n) is not an increasing sequence for all n. Instead, it's an oscillating sequence that increases from even-indexed terms to odd-indexed terms and decreases from odd-indexed terms to even-indexed terms.

To verify, let's calculate a few terms:

$$b_0 = -\frac{1}{5} \quad (451)$$

$$b_1 = \frac{1}{10} \quad (452)$$

$$b_2 = -\frac{1}{20} \quad (453)$$

$$b_3 = \frac{1}{40} \quad (454)$$

We can see that $b_0 < b_1$, $b_1 > b_2$, and $b_2 < b_3$, confirming our analysis.

Since the sequence oscillates, it cannot be an increasing sequence for all n.

Answer: Part (a): $a_1 = \frac{4}{3}$

Part (b): We've shown that $a_n > 0$ for all $n \in \mathbb{N}$ using mathematical induction.

Part (c): The limit of the sequence $\{a_n\}$ is $l = 1$.

Part (d): The sequence $\{b_n\}$ is a geometric progression with common ratio $r = -\frac{1}{2}$, which converges to 0 as $n \to \infty$.

Part (e): The sequence $\{b_n\}$ can be expressed as $b_n = -\frac{1}{5} \cdot \left(-\frac{1}{2}\right)^n$.

Part (f): The sequence $\{b_n\}$ is not an increasing sequence. It oscillates, with $b_{n+1} > b_n$ when n is even and $b_{n+1} < b_n$ when n is odd.

10 Rational Function Analysis

Problem Statement

A function f is defined by $f(x) = \frac{2x^2+x+1}{x^2-1}$

a) State the domain of f (1 mark)

b) State the coordinates of the intercepts of $y = f(x)$ (1 mark)

c) Find the asymptotes to the curve $y = f(x)$ (3 marks)

d) Find the critical values of the curve $y = f(x)$ (2 marks)

The function $f(x)$ is scaled by a factor to achieve another function $g(x)$, with curve represented as seen in the given graph.
From the graph,

e) State the domain of $g(x)$ (1 mark)

f) State the equations of each of the two asymptotes of $g(x)$ (3 marks)

g) State the limiting values of $g(x)$ as x approaches the vertical asymptotes from the left and from the right (2 marks)

h) Identify the stationary points of $g(x)$ (2 marks)

Part (a): Domain of f

Solution: The function $f(x) = \frac{2x^2+x+1}{x^2-1}$ is defined for all values of x except where the denominator equals zero.

Setting the denominator equal to zero:

$$x^2 - 1 = 0 \tag{455}$$
$$x^2 = 1 \tag{456}$$
$$x = \pm 1 \tag{457}$$

Therefore, the domain of f is $\{x \in \mathbb{R} : x \neq \pm 1\}$, or written in interval notation: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Part (b): Intercepts of $y = f(x)$

Solution: ***y*-intercept:** To find the y-intercept, we substitute $x = 0$ into $f(x)$:

$$f(0) = \frac{2(0)^2 + 0 + 1}{0^2 - 1} \quad (458)$$
$$= \frac{0 + 0 + 1}{0 - 1} \quad (459)$$
$$= \frac{1}{-1} \quad (460)$$
$$= -1 \quad (461)$$

Therefore, the y-intercept is $(0, -1)$.
***x*-intercepts:** To find the x-intercepts, we set $f(x) = 0$:

$$\frac{2x^2 + x + 1}{x^2 - 1} = 0 \quad (462)$$

This implies that the numerator equals zero (assuming $x \neq \pm 1$):

$$2x^2 + x + 1 = 0 \quad (463)$$

Using the quadratic formula with $a = 2$, $b = 1$, and $c = 1$:

$$x = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} \quad (464)$$
$$= \frac{-1 \pm \sqrt{1 - 8}}{4} \quad (465)$$
$$= \frac{-1 \pm \sqrt{-7}}{4} \quad (466)$$

Since the discriminant is negative, there are no real solutions to $2x^2 + x + 1 = 0$. Therefore, the curve $y = f(x)$ has no x-intercepts.
Therefore, the only intercept of the curve $y = f(x)$ is the y-intercept at $(0, -1)$.

Part (c): Asymptotes of $y = f(x)$

Solution: Vertical Asymptotes: Vertical asymptotes occur at values of x where the denominator of $f(x)$ is zero, but the numerator is non-zero. We already found that the denominator is zero when $x = \pm 1$.

Let's check if the numerator is zero at these points:

$$\text{For } x = 1 : 2(1)^2 + 1 + 1 = 2 + 1 + 1 = 4 \neq 0 \quad (467)$$
$$\text{For } x = -1 : 2(-1)^2 + (-1) + 1 = 2 - 1 + 1 = 2 \neq 0 \quad (468)$$

Therefore, $f(x)$ has vertical asymptotes at $x = 1$ and $x = -1$.

Horizontal Asymptote: To find the horizontal asymptote, we examine the behavior of $f(x)$ as $x \to \pm\infty$.

Since the degree of the numerator (2) is equal to the degree of the denominator (2), the horizontal asymptote is given by the ratio of the coefficients of the highest power terms:

$$y = \frac{2}{1} = 2 \quad (469)$$

Therefore, $y = 2$ is a horizontal asymptote of $f(x)$.

Slant Asymptote: Since the degree of the numerator is equal to the degree of the denominator, there is no slant asymptote.

To summarize, the asymptotes of $y = f(x)$ are:

- Vertical asymptotes: $x = 1$ and $x = -1$
- Horizontal asymptote: $y = 2$

Part (d): Critical Values of $y = f(x)$

Solution: Critical values occur at points where the derivative of $f(x)$ is zero or undefined.

First, let's find the derivative of $f(x)$ using the quotient rule:

$$f'(x) = \frac{d}{dx}\left[\frac{2x^2+x+1}{x^2-1}\right] \tag{470}$$
$$= \frac{(x^2-1)\cdot\frac{d}{dx}(2x^2+x+1)-(2x^2+x+1)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \tag{471}$$
$$= \frac{(x^2-1)(4x+1)-(2x^2+x+1)(2x)}{(x^2-1)^2} \tag{472}$$
$$= \frac{(x^2-1)(4x+1)-2x(2x^2+x+1)}{(x^2-1)^2} \tag{473}$$
$$= \frac{4x^3+x^2-4x-1-4x^3-2x^2-2x}{(x^2-1)^2} \tag{474}$$
$$= \frac{-x^2-6x-1}{(x^2-1)^2} \tag{475}$$
$$\tag{476}$$

To find where $f'(x)=0$, we set the numerator equal to zero:

$$-x^2-6x-1=0 \tag{477}$$
$$x^2+6x+1=0 \tag{478}$$

Using the quadratic formula with $a=1$, $b=6$, and $c=1$:

$$x = \frac{-6\pm\sqrt{36-4}}{2} \tag{479}$$
$$= \frac{-6\pm\sqrt{32}}{2} \tag{480}$$
$$= \frac{-6\pm 4\sqrt{2}}{2} \tag{481}$$
$$= -3\pm 2\sqrt{2} \tag{482}$$

This gives us:

$$x = -3+2\sqrt{2} \approx -0.17 \tag{483}$$
$$x = -3-2\sqrt{2} \approx -5.83 \tag{484}$$

Therefore, the critical values of $f(x)$ occur at $x=-3+2\sqrt{2}$ and $x=-3-2\sqrt{2}$.

We should check if the derivative is undefined anywhere. The derivative $f'(x)$ is undefined where $(x^2-1)^2 = 0$, which occurs when $x = \pm 1$. However, these points are not in the domain of $f(x)$, so they are not critical values.

Therefore, the critical values of the curve $y = f(x)$ are $x = -3 + 2\sqrt{2}$ and $x = -3 - 2\sqrt{2}$.

Part (e): Domain of $g(x)$

Solution: From the graph, we can observe that the function $g(x)$ has a vertical asymptote at $x = -1$. The function appears to be defined everywhere else in the real number line.

Therefore, the domain of $g(x)$ is $\{x \in \mathbb{R} : x \neq -1\}$, or in interval notation: $(-\infty, -1) \cup (-1, \infty)$.

Part (f): Equations of the Asymptotes of $g(x)$

Solution: From the graph, we can identify two asymptotes:

Vertical Asymptote: The graph clearly shows a vertical asymptote at $x = -1$.

Oblique (Slant) Asymptote: Unlike function $f(x)$, the graph of $g(x)$ shows that it has an oblique asymptote rather than a horizontal one. From the graph, we can observe that this slant asymptote passes through the points $(-1, -3)$ and $(0, -1)$.

To find the equation of this oblique asymptote, we can use the point-slope form:

$$\text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-3)}{0-(-1)} = \frac{2}{1} = 2 \tag{485}$$

$$\tag{486}$$

Using the point-slope form with the point $(0, -1)$:

$$y - (-1) = 2(x - 0) \tag{487}$$
$$y + 1 = 2x \tag{488}$$
$$y = 2x - 1 \tag{489}$$

Therefore, the equations of the asymptotes of $g(x)$ are:

- Vertical asymptote: $x = -1$
- Oblique (slant) asymptote: $y = 2x - 1$

Part (g): Limiting Values as x Approaches Vertical Asymptotes

Solution: From the graph, we can observe the behavior of $g(x)$ as x approaches the vertical asymptote at $x = -1$ from both sides.

As x approaches $x = -1$:

From the left (i.e., $x \to -1^-$): The graph shows that as x approaches -1 from values less than -1, the curve drops steeply downward. Therefore:

$$\lim_{x \to -1^-} g(x) = -\infty \tag{490}$$

From the right (i.e., $x \to -1^+$): The graph shows that as x approaches -1 from values greater than -1, the curve rises steeply upward. Therefore:

$$\lim_{x \to -1^+} g(x) = +\infty \tag{491}$$

Therefore, the limiting values of $g(x)$ as x approaches the vertical asymptote are:

- $\lim_{x \to -1^-} g(x) = -\infty$
- $\lim_{x \to -1^+} g(x) = +\infty$

Part (h): Stationary Points of $g(x)$

Solution: From the graph, we can identify two stationary points (where the derivative equals zero, resulting in horizontal tangents):

1. A local minimum appears to occur at the point $(0, -\frac{1}{2})$ 2. A local maximum appears to occur at the point $(-2, -7)$

At these points, the derivative of $g(x)$ is zero, and the function transitions from increasing to decreasing (at the local maximum) or from decreasing to increasing (at the local minimum).

Therefore, the stationary points of $g(x)$ based on the graph are:

- Local minimum: $(0, -\frac{1}{2})$
- Local maximum: $(-2, -7)$

Answer: **Part (a):** The domain of f is $\{x \in \mathbb{R} : x \neq \pm 1\}$, or $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Part (b): The only intercept of $y = f(x)$ is the y-intercept at $(0, -1)$.

Part (c): The asymptotes of $y = f(x)$ are:

- Vertical asymptotes: $x = -1$ and $x = 1$
- Horizontal asymptote: $y = 2$

Part (d): The critical values of $y = f(x)$ occur at $x = -3 + 2\sqrt{2} \approx -0.17$ and $x = -3 - 2\sqrt{2} \approx -5.83$.

Part (e): The domain of $g(x)$ is $\{x \in \mathbb{R} : x \neq -1\}$, or $(-\infty, -1) \cup (-1, \infty)$.

Part (f): The equations of the asymptotes of $g(x)$ are:

- Vertical asymptote: $x = -1$
- Oblique (slant) asymptote: $y = 2x - 1$

Part (g): The limiting values of $g(x)$ as x approaches the vertical asymptote are:

- $\lim_{x \to -1^-} g(x) = -\infty$
- $\lim_{x \to -1^+} g(x) = +\infty$

Part (h): The stationary points of $g(x)$ are:

- Local minimum: $(0, -\frac{1}{2})$
- Local maximum: $(-2, -7)$