GENERAL CERTIFICATE OF EDUCATION BOARD

General Certificate of Education Examination

JUNE 2025 **ADVANCED LEVEL**

Subject Title	**Further Mathematics**
Paper No.	**Paper 2**
Subject Code No.	**0775**

Duration: Three Hours

Answer ALL 10 questions.

For your guidance, the approximate mark allocation for parts of each question is indicated.

Mathematical formulae and tables published by the Board, and noiseless non-programmeable electronic calculators are allowed.

In calculations, you are advised to show all the steps in your working, giving your answer at each stage.

1. a) Given that $x = uy$, where u is a function of x, $u \neq 0$, show that

$$\frac{dy}{dx} = \frac{1}{u^2}\left(u - x\frac{du}{dx}\right).$$

(2 marks)

Hence, use the substitution $x = uy$ to transform the differential equation

$$x^3\frac{dy}{dx} + (4x+1)y^2 = 3x^2y$$

into a differential equation involving u and x. **(3 marks)**

b) Given the differential equation

$$\frac{d^2y}{dx^2} - 9y = e^{3x}.$$

Find the

i) complementary function, **(2 marks)**

ii) particular integral of the differential equation. **(3 marks)**

2. Given two vectors

$$\mathbf{a} = x\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$
$$\mathbf{b} = \mathbf{i} + \mathbf{j} - 2y\,\mathbf{k}, \quad x, y \in \mathbb{Z},$$

and that $\mathbf{a} \times \mathbf{b} = -\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$.

i) Find the values of the real constants, x and y. **(3 marks)**

ii) Show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. **(2 marks)**

iii) Find the Cartesian equation of the plane that contains $\mathbf{a}$ and $\mathbf{b}$ and passes through the point with position vector $\mathbf{i} - \mathbf{j}$. **(4 marks)**

3. Solve the equation

$$\cosh(\ln x) - \sinh\left(\ln\left(\tfrac{1}{2}x\right)\right) = \frac{7}{4}, \quad x > 0.$$

(6 marks)

4. Given that

$$f(x) = \frac{4x}{(x-1)(x+1)(x^2+1)}, \quad x \neq \pm 1,$$

express $f(x)$ in partial fractions. **(4 marks)**

Hence, show that

$$\int_2^3 f(x)\,dx = \ln\left(\frac{4}{3}\right).$$

(4 marks)

5. a) Show that the set of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \text{ where } a, b \in \mathbb{R},\ a \neq 0,$$

forms a group under matrix multiplication (*Assume Associativity*). **(5 marks)**

b) Solve the linear congruence

$$2x + 8 \equiv 5 \pmod{11},$$

giving your answer in the form $x = p\lambda + q$, where $p, q \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$. **(5 marks)**

6. Show that the curve with polar coordinates (r, θ) where,

$$r^2 = \frac{2}{1 - 3\sin^2\theta},$$

represents a hyperbola. **(3 marks)**

Find in polar form, the equations of the asymptotes to this hyperbola. **(5 marks)**

7. a) Solve completely the complex equation

$$z^3 = -27i.$$

(4 marks)

b) Find the centre and scale factor of the transformation described by

$$w = 3z + 2 - i.$$

(3 marks)

8. a) The parametric equations of a curve are

$$x = a(\tan\theta - \theta) \text{ and } y = a\ln\sec\theta,\ 0 < \theta < \frac{\pi}{3},$$

where a is a real constant.

Find the length of arc of this curve. **(5 marks)**

b) A transformation T is defined by matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

i) Find the determinant of $\mathbf{M}$. **(2 marks)**

ii) Find the invariant point under T. **(2 marks)**

iii) Show that the image of the line

$L : x = 2y$ is the line $L' : 9x = 4y$. **(2 marks)**

iv) Hence or otherwise, find the angle of rotation under T. **(1 mark)**

9. a) A sequence, (u_n), is defined by

$$u_0 = 3, \qquad u_{n+1} = \tfrac{1}{4}u_n + 3, \quad \forall n \in \mathbb{N}.$$

Consider another sequence, (v_n), defined by

$$v_n = u_n - 4.$$

i) Find (v_{n+1}) in terms of (v_n). **(3 marks)**

ii) Hence, deduce that (v_n) is a geometric progression and state its common ratio. **(2 marks)**

iii) Find expressions for (v_n) and also for (u_n) in terms of n. **(4 marks)**

iv) Show that (u_n) is an increasing sequence. **(3 marks)**

b) Find the radius of convergence of the series,

$$\sum_{n=0}^{\infty} \frac{5^n}{n^2 + 1} x^n.$$

(3 marks)

10. Given the function f, where,

$$f\left(x\right)=\frac{\ln x}{x-1},$$

i) find the domain of f. **(2 marks)**

ii) Evaluate

$$\lim_{x\to 0^+} f(x),$$

$$\lim_{x\to 1} f(x) \text{ and}$$

$$\lim_{x\to \infty} f(x).$$

(4 marks)

Hence or otherwise,

iii) find the asymptotes to the curve $y=f\left(x\right)$. **(2 marks)**

iv) redefine f so that it is continuous at $x=1$. **(2 marks)**

Furthermore, using the relationship $\frac{1}{x}+\ln x\geq 1$,

v) investigate f and draw its variation table. **(3 marks)**

vi) Sketch the curve of $y=f\left(x\right)$. **(2 marks)**